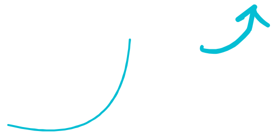
Calculus 11/9

<u>Def</u> (Def 4.6.1, 4.6.2) (Assume f is differentiable)

The graph of f is said to be

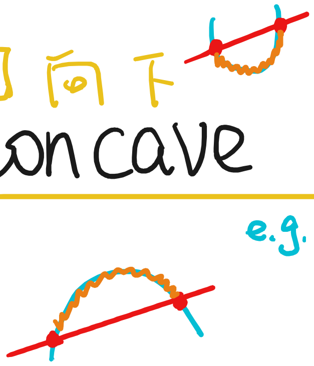
- 凹向上 <u>Concave up</u> if $f'(x)$ is increasing



$f(x)=x^2 \Rightarrow f'(x)=2x$

$\Rightarrow f''(x)=2>0$

- 凹向下 <u>Concave down</u> if $f'(x)$ is decreasing



e.g. $f(x)=-x^2$

$\Rightarrow f'(x)=-2x$

$\Rightarrow f''(x)=-2<0$

反曲點

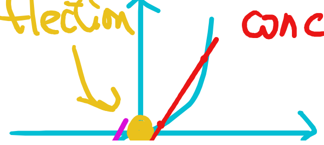
A point $(c, f(c))$ is a <u>point of inflection</u> if $\exists\ \delta>0$ s.t. f is concave up/down on $(c-\delta, c)$ and concave down/up on $(c, c+\delta)$

opposite

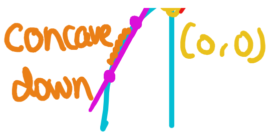
i.e. f has the opposite concavity on $(c-\delta, c)$ and $(c, c+\delta)$

e.g.

$f(x)=x^3$



$\Rightarrow f'(x) = 3x^2$



$\Rightarrow f''(x) = 6x \quad , \quad f''(0) = 0$

<u>Thm</u> (Thm 4.6.3, 4.6.4)

(i) If $(c, f(c))$ is a point of inflection, then $f''(c) = 0$ or $f''(c)$ doesn't exist

(ii) Suppose f is twice differentiable.

Then

(a) $f''(x) > 0 \quad \forall x \in (a,b)$

$\Rightarrow$ the graph of f is concave up on (a,b).

(b) $f''(x) < 0 \quad \forall x \in (a,b)$

$\Rightarrow$ the graph of f is concave down on (a,b)

Example

Example

Analyze the graph of the following function : $f(1)=5$, $f(2)=3$, $f(3)=1$

$$f(x)=x^3-6x^2+9x+1$$

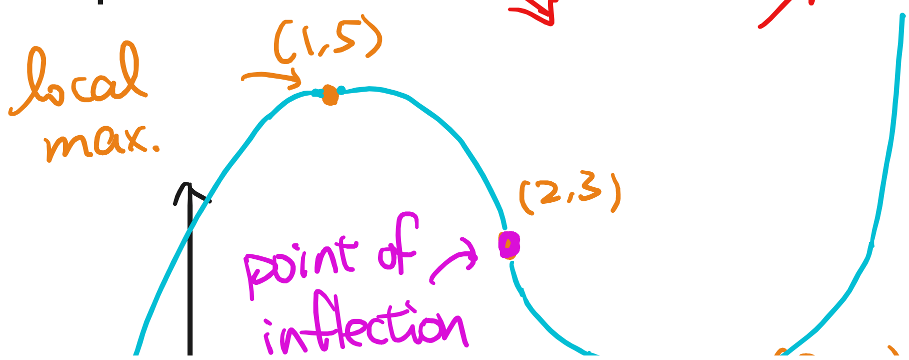
$$\Rightarrow f'(x) = 3x^2-12x+9 = \underline{3(x-1)(x-3)}$$

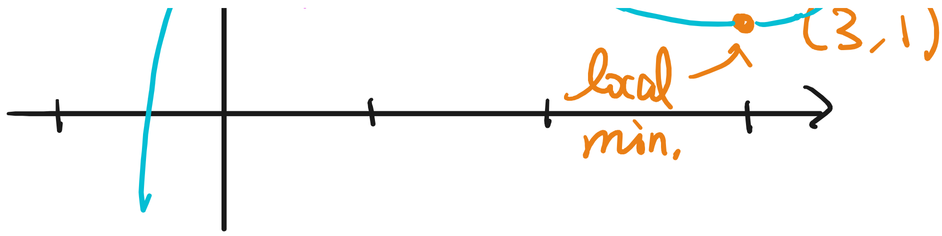
$$(f'(x)=0 \iff x=1,\ 3)$$

$$f''(x) = 6x-12 = 6(x-2)$$

$$(f''(x)=0 \iff x=2)$$

x		1		2		3	
f'	$+$ ↘		$-$ ↘		$-$ ↗		$+$ ↗
f''	$-$		$-$		$+$		$+$
f	↗		↘		↘		↗





Application of derivatives to limits

Recall

We considered

$$L = \lim_{x \to c} \frac{f(x)}{g(x)}$$

- $\lim_{x\to c} g(x) \neq 0 \Rightarrow L = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
- $\lim_{x\to c} g(x) = 0$, $\lim_{x\to c} f(x) \neq 0$ $\Rightarrow$ L doesn't exist
- $\lim_{x\to} g(x) = \lim_{x\to c} f(x) = 0 \Rightarrow$??

Thm (L'Hôpital's rule of type $\frac{0}{0}$. §11.5)

Suppose
$$\lim_{x\to c} f(x) = \lim_{x \to c} g(x) = 0$$
and f and g are differentiable near c.

If $\lim \dfrac{f'(x)}{\ \ }$ = L exists, then

$$\lim_{x\to c}\frac{}{g'(x)}$$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = L.$$

<u>Thm</u> (L'Hôpital's rule of type $\frac{\pm\infty}{\pm\infty}$, §11.6)

Suppose

$$\lim_{x\to c} f(x) = \pm\infty, \quad \lim_{x\to c} g(x) = \pm\infty$$

and f, g are differentiable near c.

If $\displaystyle\lim_{x\to c}\frac{f'(x)}{g'(x)} = L$ exists, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = L$$

<u>Example</u>

① $$\lim_{x\to\frac{\pi}{2}}\frac{\overset{\to 0}{\cos x}}{\underset{\to 0}{\pi - 2x}} = \lim_{x\to\frac{\pi}{2}}\frac{\overset{-\sin x}{\overset{\parallel}{(\cos x)'}}}{\underset{= -2 \neq 0}{(\pi-2x)'}}$$

"$\frac{0}{0}$"

$$= \frac{-\sin(\frac{\pi}{2})}{2} = \frac{1}{2} \quad \#$$

-2

"$\frac{0}{0}$"

② $$\lim_{x\to 1}\frac{\overset{\to 0}{x^4+2x^3-2x^2-1}}{\underset{\to 0}{x^3-1}} = \lim_{x\to 1}\frac{4x^3+6x^2-4x}{\underset{\to 3 \neq 0}{3x^2}}$$

$$= \frac{4+6-4}{3} = 2 \quad \#$$

"$\frac{\infty}{\infty}$"

③ $$\lim_{x\to(\frac{\pi}{2})^-}\frac{\overset{\to +\infty}{\sec x}}{\underset{\to +\infty}{1+\tan x}} = \lim_{x\to(\frac{\pi}{2})^-}\frac{\overset{\frac{\sin x}{\cancel{\cos x}}}{\tan x}\cdot\cancel{\sec x}}{\sec^{\cancel{2}}x} \qquad \sec x = \frac{1}{\cancel{\cos x}}$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = -\frac{-\sin x}{\cos^2 x}$$

$$= \tan x\cdot\sec x$$

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

$g = \cos x$

$g' = -\sin x$

$$= \lim_{x\to(\frac{\pi}{2})^-}\sin x = 1 \quad \#$$

"$0\cdot\infty$" $\to$ "$\frac{0}{0}$"

④ $$\lim_{x\to(\frac{\pi}{2})^-}\underset{\to 0}{\left(\frac{\pi}{2}-x\right)}\cdot\overset{\to +\infty}{\boxed{\tan x}}$$

$$\underset{\to 0}{\frac{\pi}{2}-x}$$

$$= \lim_{x \to (\frac{\pi}{2})^-} \frac{}{\frac{1}{\tan x}} \to 0$$

product rule

$$= \lim_{x \to (\frac{\pi}{2})^-} \frac{(\frac{\pi}{2} - x) \cdot \sin x}{\cos x} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\overset{-1}{(-1)} \cdot \sin x + (\frac{\pi}{2} - x)(\cos x)}{\underbrace{-\sin x}_{\to -1}}$$

$$= 1 \quad \#$$

⑤ " $\infty - \infty$ " $\to$ " $\frac{0}{0}$ "

$$\lim_{x \to 0} \left(\underbrace{\frac{1}{\sin x}}_{\infty} - \overset{\to \infty}{\frac{1}{x}} \right) = \lim_{x \to 0} \frac{\overset{\to 0}{x - \sin x}}{\underbrace{x \sin x}_{\to 0}}$$

$$= \lim_{x \to 0} \frac{\overset{\to 1-1=0}{1 - \cos x}}{\underbrace{\sin x + x \cos x}_{0 + 0 \cdot 1 = 0}}$$

$$= \lim_{x \to 0} \frac{\overset{\to 0}{\sin x}}{\underbrace{\cos x + \cos x + x(-\sin x)}_{1 + 1 + 0 = 2 \neq 0}}$$

$$= \frac{0}{2} = 0 \quad \#$$

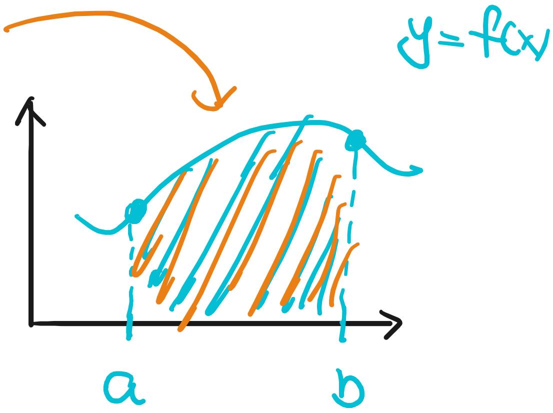
---

§ Integration

The integral of a function f from a to b, denoted by

$$\int_a^b f(x)\, dx$$

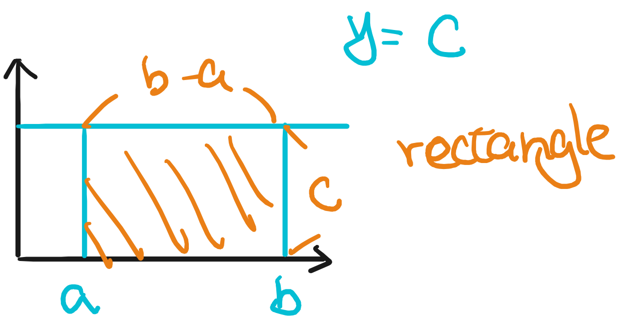
is the area



Example

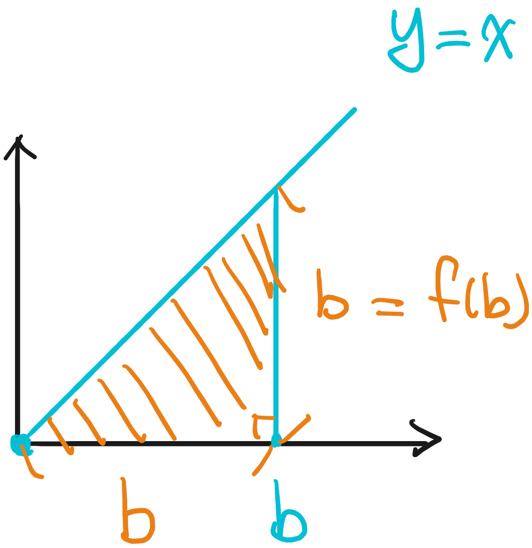
① If $f(x) = c = $ Constant, then

$$\int_a^b c \, dx = c \cdot (b-a)$$

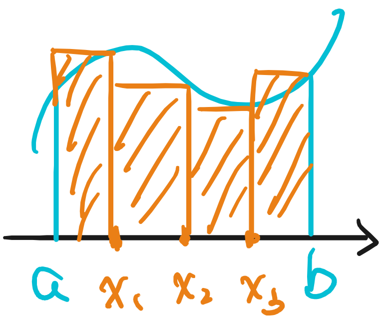


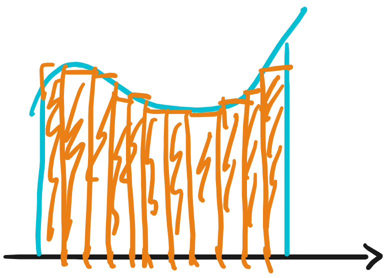
② If $f(x) = x$, then

$$\int_0^b x \cdot dx = \frac{1}{2} b \cdot b = \frac{b^2}{2}$$



§ Approximation of integrals





Def (5.2.1)

A partition P of $[a,b]$ is a finite sequence $P=\{x_0, x_1, \dots, x_n\}$, where

$$a=x_0<x_1<x_2<\cdots<x_n=b$$

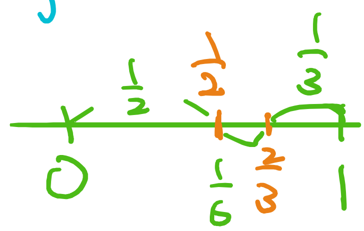
The number

$$\|P\| = \max\{|x_1-x_0|, |x_2-x_1|, \cdots, |x_n-x_{n-1}|\}$$

is called the norm of P

e.g.

$P=\{0, \frac{1}{2}, \frac{2}{3}, 1\}$ is a partition of $[0,1]$



$$\|P\| = \frac{1}{2}$$

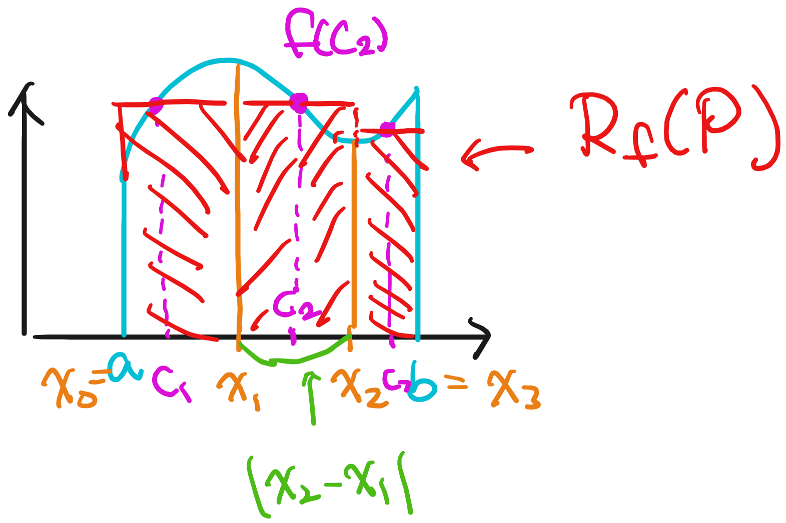
Def (5.2.6)

A Riemann sum for f on $[a,b]$

A Riemann sum for f on $[a,b]$
with respect to a partition P is

$$R_f(P) = \sum_{i=1}^{n} \underbrace{f(c_i)\cdot |x_i - x_{i-1}|}$$

where $c_i \in [x_{i-1}, x_i]$



<u>Def</u> (5.2.7)

A function f is said to be
可積分
(Riemann) integrable on $[a,b]$ if
the limite

$$\lim_{\|P\|\to 0} R_f(P)$$

exists and doesn't depend on

the choice of Riemann sums.

In this case, the number

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R_f(P)$$

is called the (definite) integral 定積分 of f from a to b

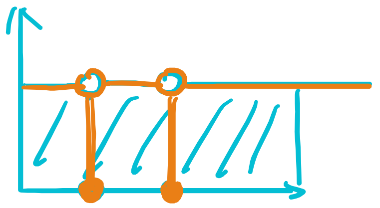
<u>Thm</u>

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

<u>Remark</u>

The values of f at finite points do NOT affect integrability and the integral $\int_a^b f(x)\,dx$.

e.g. $$f(x) := \begin{cases} 1 & \text{if } x \neq \frac{1}{2}, 1 \\ 0 & \text{if } x = \frac{1}{2}, 1 \end{cases}$$



$\int_0^2 f(x)\,dx$ exists and

$$= \int_0^2 1\,dx = 2 \cdot 1 = 2$$

#

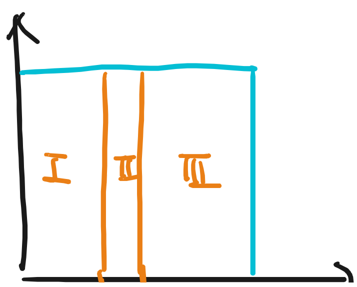
Example

Let $f(x) = 1 \quad \forall x$.

Let $P = \{\underset{x_0}{0}, \underset{x_1}{\frac{1}{3}}, \underset{x_2}{\frac{1}{2}}, \underset{x_3}{1}\}$ — partion of $[0,1]$

$$\Rightarrow \quad R_f(P) = \sum_{i=1}^{3} \underbrace{f(c_i)}_{} \overset{1}{\underset{\|}{}} \cdot (x_i - x_{i-1})$$

$$= \underset{\text{I}}{1 \cdot (\tfrac{1}{3} - 0)} + \underset{\text{II}}{1 \cdot (\tfrac{1}{2} - \tfrac{1}{3})} + \underset{\text{III}}{1 \cdot (1 - \tfrac{1}{2})}$$



$$= 1$$

In fact, for any choice of Riemann sum

choice of Riemann sum,

$$R_f(P) = 1$$

So $$\int_0^1 1\,dx = \lim_{\|P\|\to 0} R_f(P)$$

$$= 1 \qquad \#$$