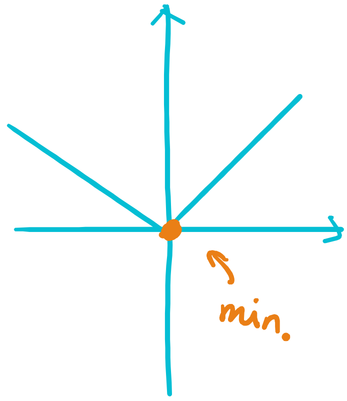
Calculus 11/7

Example

① $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



$f'(x) = \begin{cases} 1 & x > 0 \\ \text{doesn't exist} & x = 0 \\ -1 & x < 0 \end{cases}$

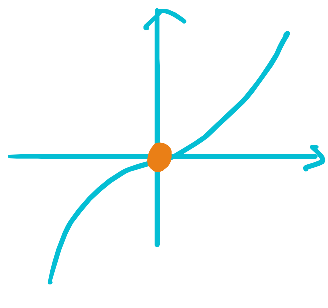
critical point: $x = 0$

x		0	
f'	$-$		$+$
f	$\searrow$	local min	$\nearrow$

② $f(x) = x^3$

$f'(x) = 3x^2$, $f'(x) = 0 \Leftrightarrow x = 0$

critical point: $x = 0$



x		0	
f'	$+$		$+$

f ↗ • ↗

neither local max nor local min

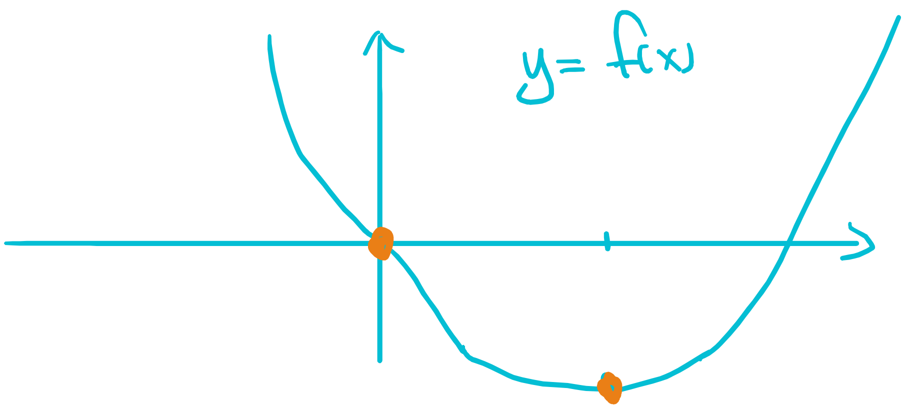
③ $f(x) = x^4 - 2x^3$

$\Rightarrow f'(x) = 4x^3 - 6x^2 = \underline{2x^2(2x-3)}$

So $f'(x) = 0 \iff x = 0, \frac{3}{2}$

Critical points: $0, \frac{3}{2}$

x		0		$\frac{3}{2}$	
f'	$-$		$-$		$+$
f	↘	•	↘	•	↗
		neither		local min.	



④ $f(x) = 2\,x^{5/3} + 5\,x^{2/3}$

$$\Rightarrow f'(x) = \frac{10}{3}x^{\frac{2}{3}} + \frac{10}{3}x^{-\frac{1}{3}}$$

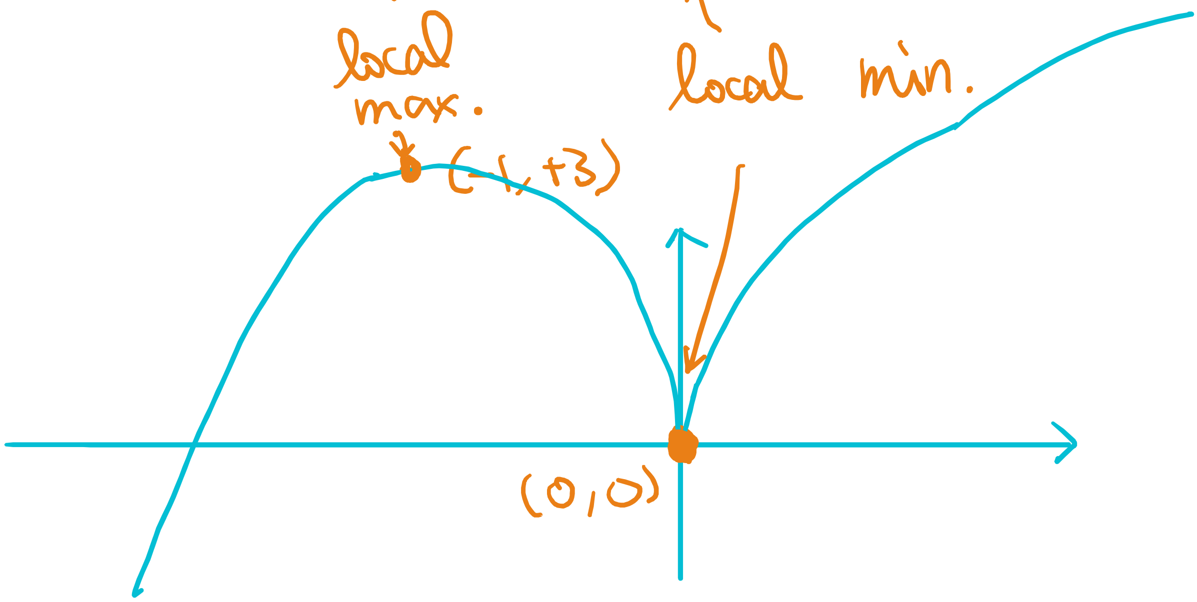
$$= \frac{10}{3\sqrt[3]{x}}(x+1)$$

doesn't exist if $x=0$

$= 0$ if $x=-1$

$\Rightarrow$ Critical points : $x = 0, -1$

x		-1		0	
f'	$+$		$-$		$+$
f	$\nearrow$	•	$\searrow$	•	$\nearrow$
		local max.		local min.	



<u>Def</u> (Def 4.4.1) We say:

- a, b are endpoints of the interval $[a,b]$
- a is an endpoint of $[a,b)$, but b is not
- b is an endpoint of $(a,b]$, but a is not

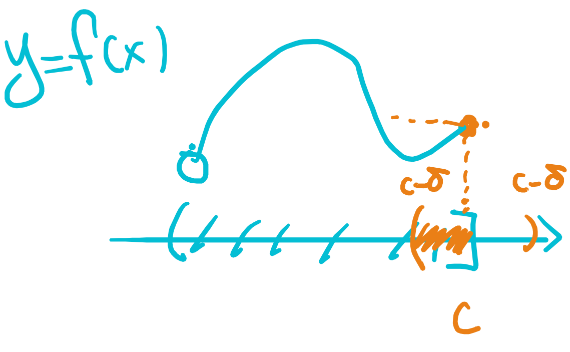
Let c be an endpoint of domain(f).

endpoint minimum local minimum

We say that f has an endpoint maximum (or local maximum) at c if $\exists\ \delta > 0$ s.t.

$$f(c) \leq f(x)$$

$$f(c) \geq f(x) \quad \forall\ x \in (c-\delta, c+\delta) \cap \text{domain}(f)$$



Recall

absolute

A function f has an absolute

minimum global minimum

maximum (or global maximum)

at d if

$$f(d) \geq f(x) \quad \forall x \in \text{domain}(f)$$

$$f(d) \leq f(x)$$

Next question:

How can we find global maximum(s)/minimum(s) of a function?

Example

① (i) Find the critical points of

$$f(x) = 1 + 4x^2 - \frac{1}{2}x^4, \qquad x \in [-1, 3]$$

and (ii) classify the extreme values

and classify the extreme values

<u>sol</u>

To find extreme values:

<u>Step1</u>: Find critical points and endpoints

- endpoints: -1 , 3

- critical points:

$$f'(x) = 8x - 2x^3 = 2x(4-x^2)$$

$$= 2x(2-x)(2+x)$$

$$f'(x) = 0 \iff x = 0, \cancel{-2}, 2$$

$-2 \notin [-1,3]$

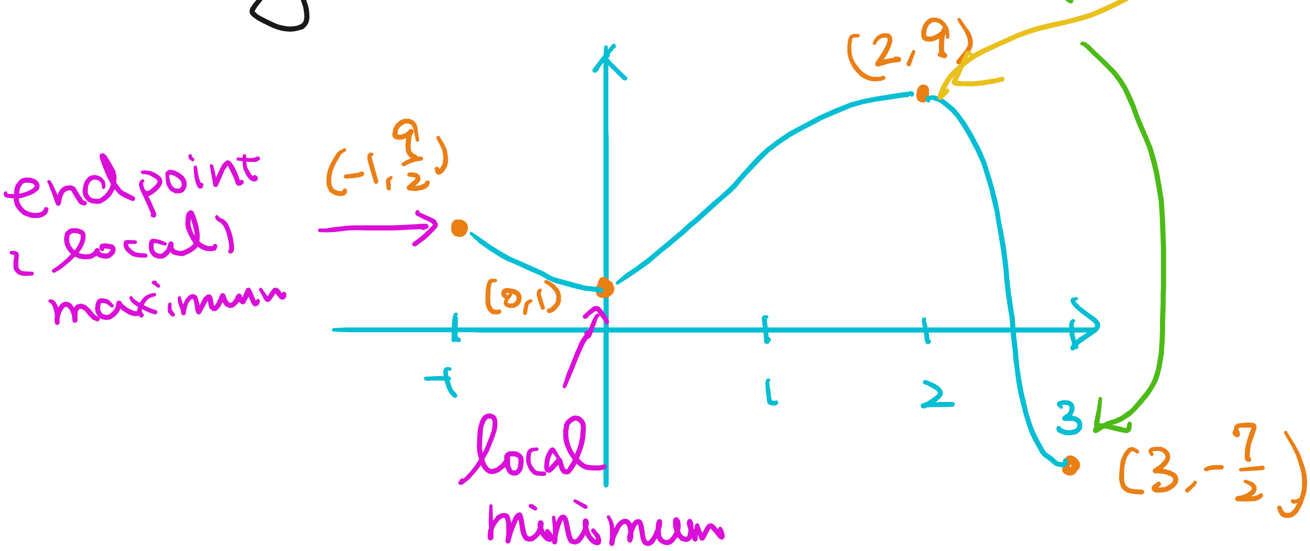
<u>Step2</u>: Compare $f(-1)$, $f(0)$, $f(2)$, $f(3)$

$$f(-1) = \frac{9}{2}, \quad f(0) = 1$$

$$f(2) = 9, \quad f(3) = -\frac{7}{2}$$

$\Rightarrow$ global max: $9 = f(2)$

global min : $-\frac{7}{2} = f(3)$



② $f(x) = \frac{1}{4}(x^3 - \frac{3}{2}x^2 - 6x + 2)$,

$x \in [-2, \infty)$

Classify extreme values.

<u>sol</u>

<u>Step1</u>

- endpoint : -2
- critical point :

$f'(x) = \frac{3}{4}x^2 - \frac{3}{4}x - \frac{6}{4}$

$= \frac{3}{4}(x^2 - x - 2) = \frac{3}{4}(x-2)(x+1)$

$f'(x) = 0 \iff x = -1, 2$

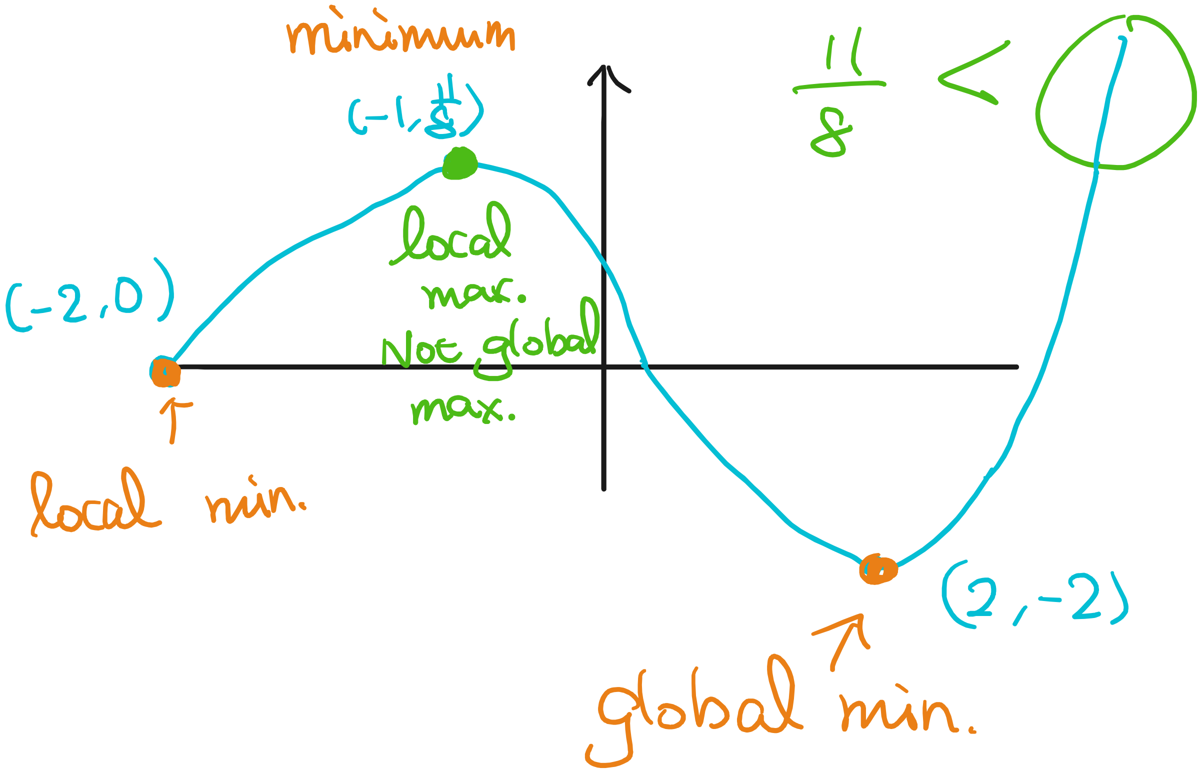
Step 2:

$f(-2) = 0, \quad f(-1) = \frac{11}{8}, \quad f(2) = -2$

x	-2		-1		2	
f'		$+$		$-$		$+$
f		↗		↘		↗

local maximum

local minimum



Conclusion:

- local min. at $x = -2, 2$

- global min. at $x = 2$
- local max. at $x = -1$
- global max. doesn't exist.

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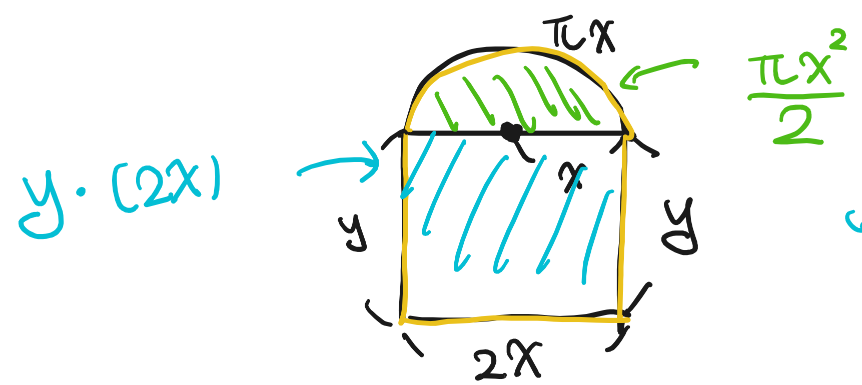
③ Consider window of the shape



(i) perimeter = 10m
(周長)

Choose x, y so that (ii) the window admits the most light (largest area)

<u>Sol</u>

Perimeter $= 10 = 2y + (2+\pi)x$

$\Rightarrow y = \frac{1}{2}(10-(2+\pi)x)$

Let

A area of window

$A(x) = A =$ area of window

$$= 2x\underline{y} + \frac{\pi}{2}x^2$$

$$= x\left(10 - (2+\pi)x\right) + \frac{\pi}{2}x^2$$

$x \geq 0$, $\boxed{\begin{array}{l} y \geq 0 \\ \quad \shortparallel \\ \frac{1}{2}\left(10 - (2+\pi)x\right) \end{array}}$ $\Longleftrightarrow$ $x \leq \dfrac{10}{2+\pi}$

$x \in \left[0, \dfrac{10}{2+\pi}\right]$

endpoint: $x = 0, \dfrac{10}{2+\pi}$

critical:

$$A'(x) = 10 - (2+\pi)\cdot 2x + \pi x$$

$$= \underline{10 - (4+\pi)x}$$

$$A'(x) = 0 \Longleftrightarrow x = \frac{10}{4+\pi} \in \left[0, \frac{10}{2+\pi}\right]$$

x	0		$\frac{10}{4+\pi}$		$\frac{10}{2+\pi}$
$A'(x)$		$+$		$-$	
$A(x)$		$\nearrow$		$\searrow$	

max.

The global maximum value of A is $A\left(\frac{10}{4+\pi}\right)$ i.e.

$$x = \frac{10}{4+\pi}, \quad y = \frac{1}{2}\left(10 - (2+\pi)\frac{10}{4+\pi}\right)$$

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National Tsing Hua University

Calculus I – Exam 1

Instructor: Hsuan-Yi Liao

Fall, 2023

Name: Answer

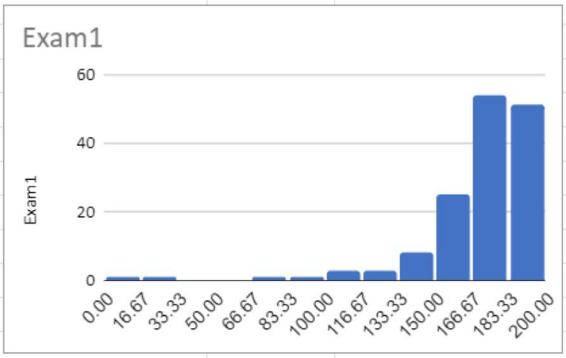
Student ID: ____________________

- This exam contains 11 pages (including this cover page) and 7 questions.
- Total of points is 200.
- Time limit: **100 minutes**.
- Write down your computation or arguments in details unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.
- The use of L'Hôspital's rule is **NOT** allowed in this exam.

Distribution of Marks

Question	Points	Score
1	16	
2	80	
3	16	
4	24	

Average	171.106
Average (except zero)	172.255
Quartile 0	0
Quartile 1	165.25
Quartile 2	178
Quartile 3	187
Quartile 4	200
標準差	27.6388
非零數	149
不到60%人數	7
滿分/[redacted]數	2



1. (16 points) State the precise definition of $\lim_{x\to c} f(x) = L$ and use it to show that

$$\lim_{x\to 3}(x-1)^2 = 4.$$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon \quad \forall\ 0 < |x - c| < \delta$

$\forall \varepsilon > 0,\ \exists \delta = \min\{\frac{\varepsilon}{5}, 1\}$ s.t.

$$|(x-1)^2 - 4| = |x-1-2| \cdot |x-1+2|$$
$$\leq |x-3| \cdot (|x-3|+4) < \frac{\varepsilon}{5} \cdot (1+4) = \varepsilon$$

$\forall\ 0 < |x-3| < \delta$ #

2. Decide whether or not the indicated limit exists. Evaluate the limits that do exist. If the limit does not exist, explain why.

   (a) (8 points) $\lim_{x\to -4}\left(\frac{2x}{x+4}+\frac{8}{x+4}\right)$. $= \lim_{x\to -4} \frac{2\cancel{(x+4)}}{\cancel{x+4}} = 2$

   (b) (8 points) $\lim_{x\to -1}\frac{1-x}{x+1}$. $1-x \to 2 \neq 0$, $x+1 \to 0$ $\Rightarrow$ doesn't exist

   (c) (8 points) $\lim_{x\to 2} f(x)$ if $f(x)=\begin{cases} 3, & x \text{ an integer} \\ x+2, & \text{otherwise.}\end{cases}$ $= 4$

   (d) (8 points) $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$. $= \lim_{x\to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}} = 4$

(e) (8 points) $\lim_{x\to 0} \frac{\tan^2(3x)}{4x^2}$. $= \lim_{x\to 0} \frac{1}{\cos^2(3x)} \cdot \frac{9}{4} \cdot \frac{(\sin 3x)^2}{(3x)^2} = \frac{9}{4}$

$\frac{\sin^2 3x}{\cos^2(3x)}$

(f) (8 points) $\lim_{x\to 0} \frac{\cos x - 1}{2x}$. $= \lim_{x\to 0} \frac{-1}{2} \cdot \frac{1-\cos x}{x} = 0$

△ (g) (8 points) $\lim_{x\to \frac{\pi}{2}} \frac{\sin 2x}{x - \frac{\pi}{2}}$.

" $y = x - \frac{\pi}{2}$

$$\lim_{y\to 0} \frac{\sin 2(y+\frac{\pi}{2})}{y} = \lim_{y\to 0} \frac{\sin 2y \cdot \underbrace{\cos\pi}_{=-1} + \cancel{\cos 2y \cdot \sin\pi}^{=0}}{y}$$

$$= \lim_{y\to 0} -2 \cdot \frac{\sin 2y}{2y} = -2 \ \#$$

cos x, sin x are continuous

(h) (8 points) $\lim_{x\to 0} \sin\left(\frac{\pi}{2}\cos(\tan x)\right)$. $= \sin\left(\frac{\pi}{2}\cos\left(\lim_{x\to 0}\tan x\right)\right) = \sin\left(\frac{\pi}{2}\cos 0\right)$

$= 1$.

(i) (8 points) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1}$.

(j) (8 points) $\lim_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{9/5} + 3x + \sqrt{x}}$.

$$= \lim_{\underset{\sim\sim}{x\to -\infty}} \frac{\left(\frac{1}{-x}\right)\cdot\sqrt{x^2+1}}{(x+1)\underbrace{\left(-\frac{1}{x}\right)}_{>0}}$$

$$= \lim_{x\to -\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{-1-\frac{1}{x}} = -1 \quad \#$$

$$-\frac{2}{15}$$

$$= \lim_{x\to\infty} \frac{2x^{\left(\frac{5}{3}-\frac{9}{5}\right)} - x^{\frac{1}{3}-\frac{9}{5}} + 7\cdot x^{-\frac{9}{5}}}{1+3x^{-\frac{4}{5}} + x^{\frac{1}{2}-\frac{9}{5}}} = 0 \quad \#$$

8 3. (16 points) Evaluate the limit $\lim_{x\to 0} \sin(x^2)\cos(\frac{1}{x})$. Prove your answer.

$$0 \le \left|\sin(x^2)\cos(\tfrac{1}{x})\right| \le |\sin(x^2)| \to 0$$

as $x \to 0$

Pinching Thm $\Rightarrow$

$$\lim_{x\to 0} \sin(x^2)\cdot\cos(\tfrac{1}{x}) = 0$$

#

4. Determine whether or not the function is continuous at the indicated point. Explain your answers.

(a) (8 points) $f(x) = x^3 - 5x + 1\,,\quad x = 2.$ Continuous. since f is a polynomial

(b) (8 points) $f(x) = \tan x\,,\quad x = \pi/2.$ discontinuous since $f(\frac{\pi}{2})$ is NOT defined

(c) (8 points) $f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & x \neq 1, \\ 0, & x = 1, \end{cases} \quad x = 1.$

discontinuous, since $\lim_{x \to 1} f(x)$ doesn't exist

5. (16 points) Show that if f is continuous on $[0,1]$ and $0 \le f(x) \le 1$ for all x in $[0,1]$, then there exists at least one point c in $[0,1]$ at which $f(c) = c$.

Let $g(x) = f(x) - x$

Since $g(x)$ is a sum of two continuous functions $f(x)$ and $-x$, the function $g(x)$ is continuous on $[0, 1]$.

Since

$$g(0) = f(0) - 0 = f(0) \ge 0$$

$$g(1) = f(1) - 1 \le 0$$

by the intermediate value thm, $\exists\, c \in [0,1]$ s.t.

$$g(c) = 0$$
$$\parallel$$
$$f(c) - c$$

$\Rightarrow \quad f(c) = c \qquad \#$

6. Differentiate the following functions.

(a) (8 points) $f(x) = \dfrac{x^3}{1-x}$.

$$f'(x) = \frac{3x^2 \cdot (1-x) - x^3 \cdot (-1)}{(1-x)^2} = \frac{-2x^3 + 3x^2}{(1-x)^2}$$

#

(b) (8 points) $f(x) = \left(1+\dfrac{1}{x}\right)\left(1+\dfrac{1}{x^2}\right)$.

$$f'(x) = \left(1+\frac{1}{x}\right)' \cdot \left(1+\frac{1}{x^2}\right) + \left(1+\frac{1}{x}\right) \cdot \left(1+\frac{1}{x^2}\right)'$$

$$= \left(\frac{-1}{x^2}\right) \cdot \left(1+\frac{1}{x^2}\right) + \left(1+\frac{1}{x}\right) \cdot \left(-2 \cdot \frac{1}{x^3}\right)$$

$$= \frac{-1}{x^2} - \frac{1}{x^4} - 2\frac{1}{x^3} - 2\frac{1}{x^4}$$

$$= -\frac{1}{x^2} - 2\frac{1}{x^3} - 3\frac{1}{x^4}$$

#

$\triangle$ (c) (8 points) $f(x) = \dfrac{x(x-1)}{1-\frac{1}{x-2}}$. $= \quad x(x-1)(x-2) \cdot \dfrac{1}{x-3}$

$$f'(x) = (x-1)(x-2)\cdot(x-3)^{-1} + x(x-2)\cdot(x-3)^{-1}$$
$$+ x(x-1)\cdot(x-3)^{-1} - x(x-2)\cdot(x-3)^{-2}$$

\#

7. True or false? (No need to explain.)

T (a) (2 points) If $\lim_{x\to c}\Big(f(x)+g(x)\Big)$ exists but $\lim_{x\to c} f(x)$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

T (b) (2 points) If $\lim_{x\to c}\Big(f(x)+g(x)\Big)$ and $\lim_{x\to c} f(x)$ exist, then $\lim_{x\to c} g(x)$ exists.

T (c) (2 points) If $\lim_{x\to c}\sqrt{f(x)}$ exists, then $\lim_{x\to c} f(x)$ exists.

F (d) (2 points) If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c}\sqrt{f(x)}$ exists.

F (e) (2 points) If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c}\dfrac{1}{f(x)}$ exists.

T (f) (2 points) If $\lim_{x\to c}\dfrac{f(x)}{g(x)}$ exists and $\lim_{x\to c} g(x)=0$, then $\lim_{x\to c} f(x)=0$.

T (g) (2 points) The function $f(x)=x^2$ attains a maximum value on $[-1,1]$.

F (h) (2 points) The function $f(x)=x^2$ attains a maximum value on $(-1,1)$.

T (i) (2 points) The equation $2x^3-4x^2+5x-4=0$ has a solution in $[1,2]$.

T (j) (2 points) The equation $\sin x+2\cos x-x^2=0$ has a solution in $[0,\pi/2]$.

T (k) (2 points) If a function f is differentiable at c, then f is continuous at c.

F (l) (2 points) If a function f is continuous at c, then f is differentiable at c.