Calculus 10/26

<u>Example</u>

Show that

$$p(x) = 2x^3 + 5x - 1 \Rightarrow p'(x) = 6x^2 + 5$$

has exactly one real root.

<u>pf</u>

<u>Step 1</u>

Note that

$$p(0) = -1 < 0$$

$$p(1) = 2+5-1 = 6 > 0.$$

Since p is continuous, by the intermediate value thm,

$\exists\ c \in (0,1)$ s.t.

$$p(c) = 0$$

That is, p has a root.

<u>Step 2</u>: p has only one root.

Assume $d \neq c$ s.t. $p(d) = 0$.

Since p is differentiable on $(-\infty, \infty)$,

by MVT, $\exists\ e$ s.t.

$$p'(e) = \frac{\overset{=0}{p(d)} - \overset{=0}{p(c)}}{d - c} = 0$$

But

$$p'(e) = \overset{\geq 0}{6e^2} + 5 > 0 \quad \neq 0$$

$$(\longrightarrow\longleftarrow)$$

So c is the only root of $p(x)$. #

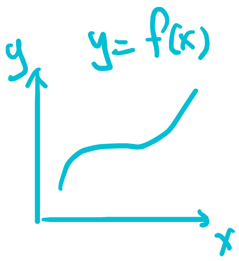
§ Monotone functions

Def (Def 4.2.1)

A function f is said to be

遞增 (= "non-decreasing" in textbook)

(i) **increasing** on (a,b) if

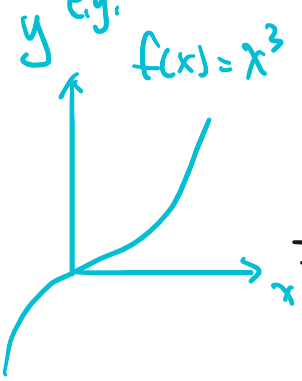


$$x_1 < x_2, \quad x_1, x_2 \in (a,b)$$

$$\Rightarrow \quad f(x_1) \le f(x_2)$$

嚴格遞增 (= "increasing" in book)

(ii) **strictly increasing** on (a,b) if

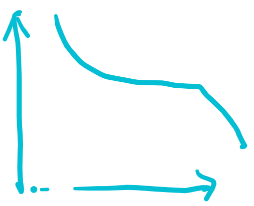


$$x_1 < x_2, \quad x_1, x_2 \in (a,b)$$

$$\Rightarrow \quad f(x_1) \lneq f(x_2)$$

(= "non-increasing" in book)

(iii) **decreasing** on (a,b) if



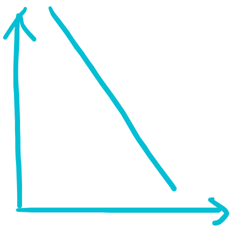
$$x_1 < x_2, \quad x_1, x_2 \in (a,b)$$

$$\Rightarrow \quad f(x_1) \ge f(x_2)$$

(= "decreasing" in book)

(iv) **strictly decreasing** on (a,b) if

$x_1 < x_2 \quad , \quad x_1, x_2 \in (a,b)$



$\Rightarrow \quad f(x_1) \gneq f(x_2)$

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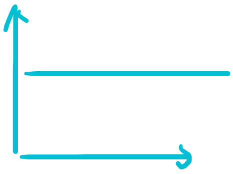
(v) Constant on (a,b) if.

$f(x_1) = f(x_2) \quad \forall \, x_1, x_2 \in (a,b)$

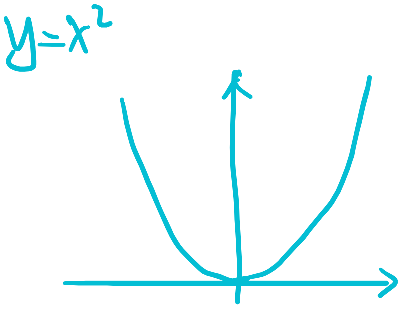


Example

① $f(x) = x^2$ is decreasing on $(-\infty, 0]$

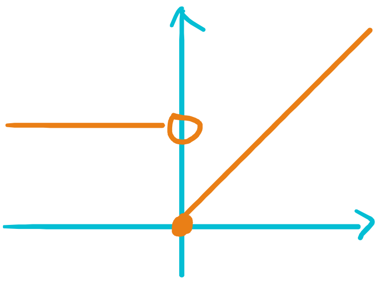
increasing on $[0, \infty)$

neither on $(-\infty, \infty)$



② $f(x) = \begin{cases} 1 & x < 0 \\ x & x \geq 0 \end{cases}$ is Constant on $(-\infty, 0)$

and increasing on $[0, \infty)$



③ $f(x) = x^3$ is increasing on $(-\infty, \infty)$

④ $f(x) = \begin{cases} 1, & x \text{ is rational (有理數)} \\ 0, & x \text{ is irrational (無理數)} \end{cases}$

is neither increasing nor decreasing.

Thm (Thm 4.2.2, Thm 4.2.3)

Suppose f is differentiable on (a,b)

Then

(i) if $f'(x) \geq 0 \quad \forall x \in (a,b)$, then f is increasing on (a,b)

(ii) if $f'(x) \gneq 0 \quad \forall \in (a,b)$, then f is strictly increasing on (a,b)

(iii) if $f'(x) \leq 0 \quad \forall x \in (a,b)$, then

f' is decreasing on (a,b)

(iv) if $f'(x) \lneq 0$ $\forall x \in (a,b)$, then
f is strictly decreasing on (a,b)

(v) if $f'(x) = 0$ $\forall x \in (a,b)$, then
f is constant on (a,b)

<u>pf</u>

(i) For $x_1 < x_2$, $x_1, x_2 \in (a,b)$,

by MVT, $\exists$ $c \in \underset{\subset (a,b)}{(x_1, x_2)}$ s.t.

$$\frac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{>0}} = f'(c) \geq 0$$

$\Rightarrow$ $f(x_2) - f(x_1) \geq 0$

(ii) ~ (v) are similar

i.e. $f(x_1) \leq f(x_2)$

$\Rightarrow$ f is increasing #

Example

Consider

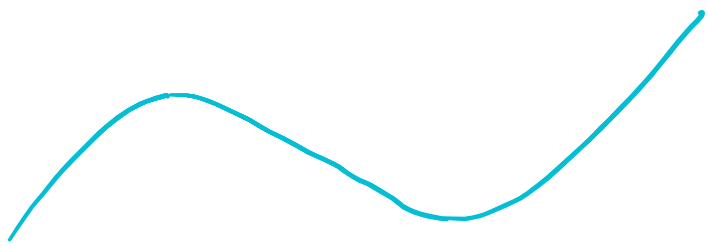
$$f(x) = x^3 - 3x$$

$a^2 - b^2 = (a-b)(a+b)$

$$\Rightarrow \quad f'(x) = 3x^2 - 3$$

$$= 3(x-1)(x+1)$$

x		-1		1	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$



About constant functions:

Thm (Thm 4.2.4)

Suppose f and g are differentiable on (a,b). Then

(i) $f'(x)=0 \quad \forall x\in(a,b)$

$$\iff \exists \underset{\text{constant}}{c} \text{ s.t. } f(x)=c \quad \forall x\in(a,b)$$

(ii) $f'(x)=g'(x) \quad \forall x\in(a,b)$

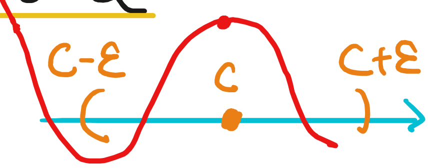
$\iff \exists$ constant C s.t.

$$f(x)=g(x)+C \quad \forall x\in(a,b)$$

§ Extreme values

<u>Def</u> (Def 4.3.1)

We say that a function f has a local maximum value at $x=c$ if $\exists\, \varepsilon>0$ s.t.



$$f(c) \geq f(x) \quad \forall x \in (c-\varepsilon, c+\varepsilon)$$

We say that f has a <u>local minimum value</u> at $x = c$ if $\exists\ \varepsilon > 0$ s.t.

$$f(c) \leq f(x) \quad \forall x \in (c-\varepsilon, c+\varepsilon)$$

<u>Remark</u>

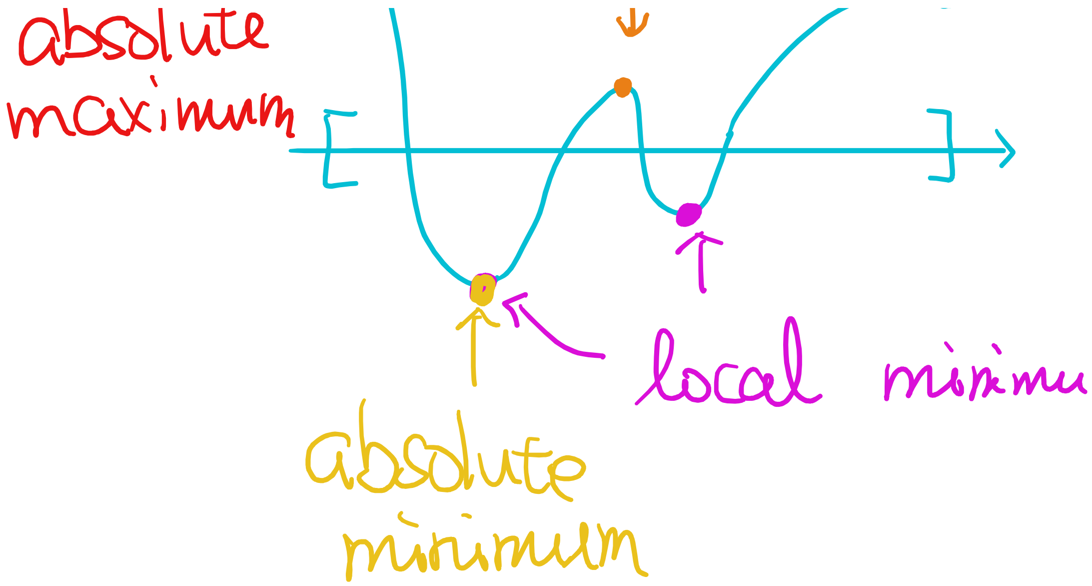
maximum = global maximum

If f has an <u>absolute maximum</u> (resp. absolute minimum) at $x = c$, then f has a local maximum at $x = c$ (resp. local minimum)

resp. = respectively

local maximum

e.g.



<u>Thm</u> (Thm 4.3.2)

If f has a local maximum/minimum at $x=c$, then either

$f'(c)=0$ or $f'(c)$ doesn't exist

pf: see the proof of the lemma on Tuesday.

<u>Def</u> (Def. 4.3.3)

A point c is called a <u>critical</u> point if either $f'(c)=0$ or

$f'(c)$ doesn't exist.

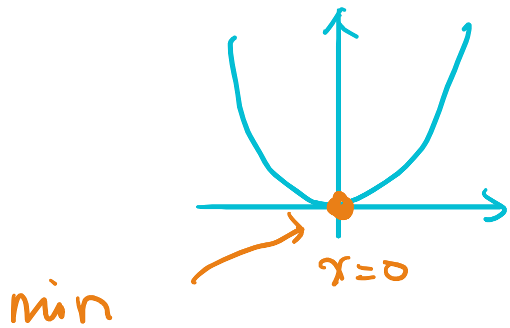
Example

Find critical points:

① $f(x) = x^2 \Rightarrow f'(x) = 2x$

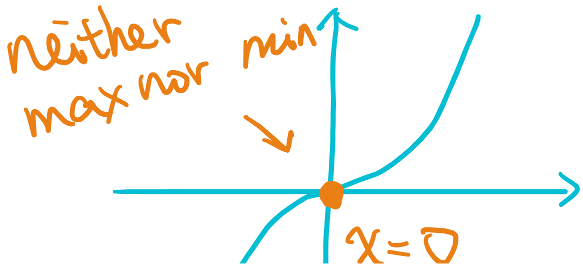
$f'(x) = 0 = 2x \Leftrightarrow x = 0$

So 0 is a critical point of f



② $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

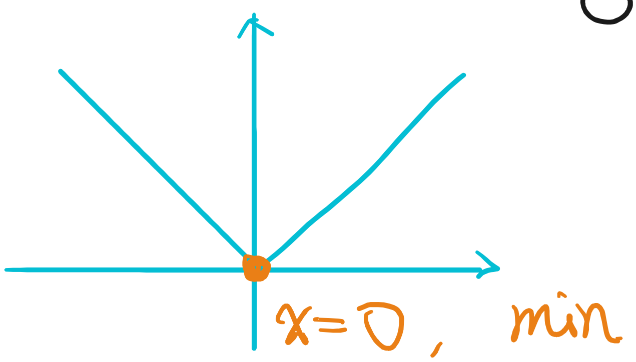
So $f'(x) = 0 = 3x^2 \Leftrightarrow x = 0$



③ $f(x) = |x|$, $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$ $\Rightarrow$ $f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

$f'(0)$ doesn't exist.

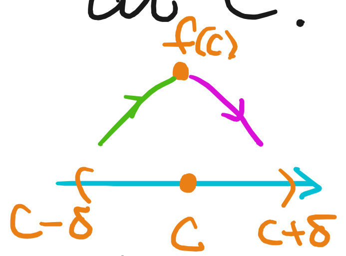
$\Rightarrow$ 0 is a critical point of $f(x) = |x|$



<u>Thm</u> (First order derivative test, Thm 4.3.4)

Suppose c is a critical point of f and f is continuous at c.

If $\exists\, \delta > 0$ s.t.



(i) $f'(x) > 0 \quad \forall x \in (c-\delta, c)$, and

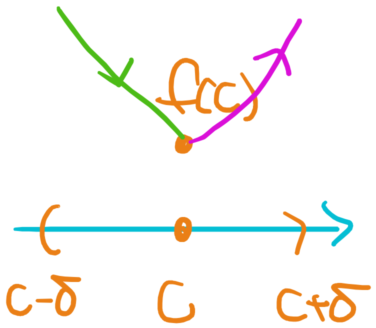
$f'(x) < 0 \quad \forall x \in (c, c+\delta)$

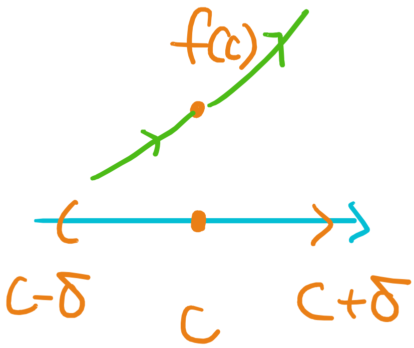
then f has a local maximum at c.

(ii) $f'(x) < 0 \quad \forall x \in (c-\delta, c)$, and

$f'(x) > 0 \quad \forall x \in (c, c+\delta)$

then f has a local minimum at c





(iii) either

$f'(x) > 0$ or $f'(x) < 0$

$\forall x \in (c-\delta, c+\delta)$, then

f does NOT have a local extreme at c.

Thm (Second order derivative test, Thm 4.3.5)

Suppose $f''(c)$ exists, $f'(c)=0$.

(i) If $f''(c)>0$, then f has a local minimum at c

(ii) If $f''(c)<0$, then f has a local maximum at c

(iii) If $f''(c)=0$, then anything can happen.

Example

$f(x)=x^2 \Rightarrow f'(x)=2x, \; f''(x)=2$

>0

x		0	
$f'(x)$	$-$	0	$+$
$f''(x)$		$+$	

$f'(x) \mid \quad \searrow \quad \mid \quad \nearrow$

local minimum

#