Calculus 10/24

<u>Recall</u>: ① chain rule

$$(f\circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

② $(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

e.g. $y = x+1$ $\qquad \frac{d(x+1)}{dx} = 1$

$$\left((x+1)^{100}\right)' = \frac{d(y^{100})}{dy} \cdot \boxed{\frac{dy}{dx}}$$

$$= 100 \cdot y^{99} = 100 \cdot (x+1)^{99}$$

#

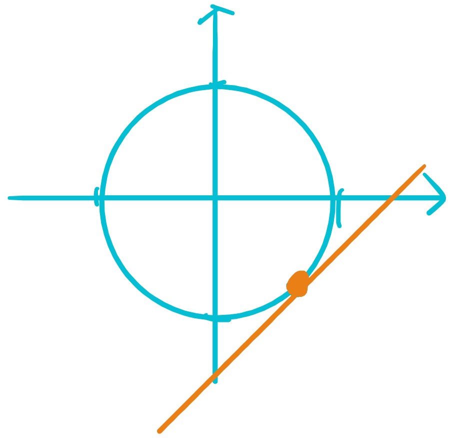
§ Implicit differentiation (§ 3.7)

Example

① Find the tangent line of

$$x^2+y^2=1$$

at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.



sol

Take derivatives of the both sides with respect to x :

$$\frac{d}{dx}(x^2+y^2) = \underbrace{\frac{d}{dx}(1)}_{=0}$$

$$\frac{d}{dx}(x^2+y^2) = \underbrace{\frac{d}{dx}(x^2)}_{=2x} + \frac{d}{dx}(y^2)$$

$$= 2x + \underbrace{\frac{d}{dy}(y^2)}_{=2y} \cdot \frac{dy}{dx}$$

// chain rule ($z=y^2$)

slope of tangent line

$$\Rightarrow \quad 2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x,y)=\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)} = -\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}$$

$$= 1$$

$\Rightarrow$ the tangent line is

$$\left(y + \frac{1}{\sqrt{2}}\right) = 1 \cdot \left(x - \frac{1}{\sqrt{2}}\right) \quad \#$$

(2) Find the tangent line of

Find the tangent

$$\circledast \quad 2x^3 + 2y^3 = 9xy$$

at $(1, 2)$.

Sol

$\circledast \Rightarrow$

$$\frac{d}{dx}\left(2x^3 + 2y^3\right) = \frac{d}{dx}(9xy)$$

$$\text{L.H.S.} = \underbrace{\frac{d}{dx}(2x^3)}_{6\cdot x^2} + \underbrace{\frac{d}{dx}(2y^3)}_{= \frac{d(2y^3)}{dy}\cdot\frac{dy}{dx}}$$

chain rule

$$= 6x^2 + 6y^2 \cdot \frac{dy}{dx}$$

product rule

$$\text{R.H.S.} = 9\underbrace{\frac{d}{dx}(xy)}_{= \left(\frac{dx}{dx}\right)\cdot y + x\cdot\frac{dy}{dx}}$$

$$= 9\left(1\cdot y + x\cdot\frac{dy}{dx}\right)$$

$$\Rightarrow \quad 6x^2 + 6y^2 \cdot \frac{dy}{dx} = 9y + 9x \frac{dy}{dx}$$

$$\Rightarrow \quad (6y^2 - 9x)\frac{dy}{dx} = 9y - 6x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{9y - 6x^2}{6y^2 - 9x} = \frac{3y - 2x^2}{2y^2 - 3x}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = \frac{3\cdot 2 - 2\cdot 1^2}{2\cdot 2^2 - 3\cdot 1}$$

$$= \frac{4}{5}$$

$\Rightarrow$ tangent line is

$$(y-2) = \frac{4}{5}\cdot(x-1)$$

#

③ Suppose

$$\cos(x-y) = xy.$$

Find $\dfrac{dy}{dx} = ?$

<u>sol</u>

product rule

$$\underbrace{\frac{d}{dx}\left(\cos(x-y)\right)} = \boxed{\frac{d}{dx}(xy)} = \frac{dx}{dx}\cdot y + x\,\frac{dy}{dx}$$

$$= y + x\cdot\frac{dy}{dx}$$

let $z = x-y$

chain rule

$$\frac{d\cos z}{dz}\cdot\frac{dz}{dx} = (-\sin z)\cdot\frac{d(x-y)}{dx}$$

$$= \left(-\sin(x-y)\right)\cdot\left(1-\frac{dy}{dx}\right)$$

So

$$-\sin(x-y) + \sin(x-y)\cdot\boxed{\frac{dy}{dx}}$$

$$= y + x\cdot\boxed{\frac{dy}{dx}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y + \sin(x-y)}{\sin(x-y) - x} \quad \#$$

Example

Find the derivatives of the following functions :

① $x^{\frac{1}{n}}$, $x > 0$

② $x^{\frac{m}{n}}$, $x > 0$

③ $\sqrt{\frac{x}{1+x^2}}$, $x > 0$

sol

① Let $y = x^{\frac{1}{n}} \Rightarrow y^n = x$

$$\Rightarrow \boxed{\frac{d}{dx}(y^n)} = \frac{d}{dx}(x) = 1$$

$$\underbrace{\qquad\qquad\qquad}_{\text{chain rule}}$$

$$\frac{d\,y^n}{dy}\cdot\frac{dy}{dx} = n\cdot y^{n-1}\cdot\frac{dy}{dx}$$

$$\Rightarrow\quad n\cdot\left(x^{\frac{1}{n}}\right)^{n-1}\cdot\boxed{\frac{dy}{dx}} = 1$$

$$\Rightarrow\quad \frac{dy}{dx} = \frac{1}{n\cdot x^{\frac{n-1}{n}}}$$

$$= \frac{1}{n}\cdot x^{-\frac{n-1}{n}}$$

$$-\frac{n-1}{n} = -\frac{n}{n}+\frac{1}{n} = \frac{1}{n}-1$$

$$\frac{d}{dx}\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\cdot x^{\frac{1}{n}-1} \qquad \#$$

② $$\frac{d}{dx}\left(x^{\frac{m}{n}}\right) = \frac{d}{dx}\left(x^{\frac{1}{n}}\right)^{m}$$

$y = x^{\frac{1}{n}}$, chain rule

$$= \frac{d\, y^{m}}{dy} \cdot \frac{dy}{dx}$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{n} \cdot x^{\frac{1}{n}-1}$

$$= m \cdot y^{m-1} \cdot \frac{1}{n} \cdot x^{\frac{1}{n}-1}$$

$$= \frac{m}{n} \cdot \left(x^{\frac{1}{n}}\right)^{m-1} \cdot x^{\frac{1}{n}-1}$$

$$= \frac{m}{n} \cdot x^{\frac{m-1}{n}+\frac{1}{n}-1}$$

$$\boxed{\frac{d\, x^{\frac{m}{n}}}{dx} = \frac{m}{n}\, x^{\frac{m}{n}-1}}$$

#

③ $$\frac{d}{dx}\sqrt{\frac{x}{1+x^2}}$$

$$= \left(\frac{x}{\ldots}\right)^{\frac{1}{2}}$$

$$= \frac{d}{dx}\left(\frac{\quad}{1+x^2} \right)$$

chain rule

$$= \frac{1}{2} \cdot \left(\frac{x}{1+x^2} \right)^{\frac{1}{2}-1} \cdot \frac{d}{dx}\left(\frac{x}{1+x^2} \right)$$

$$= \frac{1}{2} \cdot \left(\frac{x}{1+x^2} \right)^{-\frac{1}{2}} \cdot \frac{1 \cdot (1+x^2) - x \cdot 2x}{(1+x^2)^2}$$

$$= \frac{1}{2} \frac{\sqrt{1+x^2}}{\sqrt{x}} \cdot \frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{1}{2} \frac{1-x^2}{\sqrt{x} \cdot (1+x^2)^{\frac{3}{2}}} \qquad \#$$

§ Applications of derivatives

(均值定理)

§ Mean value theorem

Lemma

Lemma

Let f be a differentiable function on (a,b)

Suppose f takes a maximum value or a minimum value at $c \in (a b)$.

Then $f'(c) = 0$

pf

Assume f takes a maximum value at $x = c$. (The case of minimum is similar.)

Since

$$f(c+h) - f(c) \leq 0 \quad \forall h \in (a-c, b-c)$$

we have

$\Downarrow$ $c+h \in (a,b)$

① $$\frac{f(c+h)-f(c)}{h} \le 0, \text{ for } h>0$$

② $$\frac{f(c+h)-f(c)}{h} \ge 0, \text{ for } h<0$$

① $\Rightarrow$ $$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$$

② $\Rightarrow$ $$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \ge 0$$

Since f is differentiable,

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c) \text{ exists}$$

$$= \underline{\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}} = \underline{\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}}$$

$0 \geq$ $\Rightarrow 0$

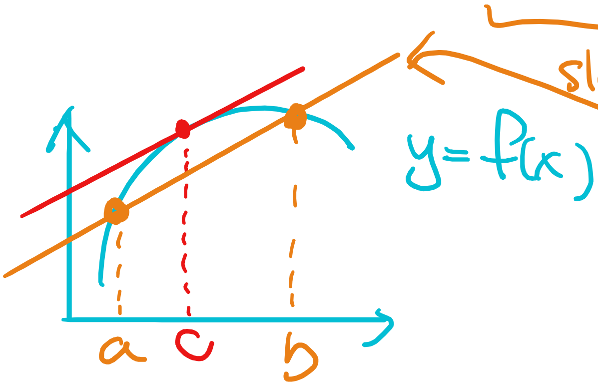
$$\Rightarrow \quad 0 \leq f'(c) \leq 0$$

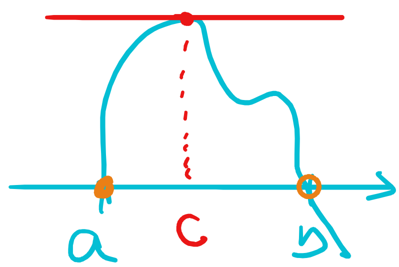
$$\Rightarrow \quad f'(c) = 0 \quad \#$$

<u>Mean Value Thm</u> (Thm 4.1.1)

If f is differentiable on (a,b) and continuous on $[a,b]$, then there exists $c \in (a,b)$ s.t.

$$f'(c) = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of}}$$





In particular (Rolle's Thm. Thm 4.1.3)

if in addition $f(a)=f(b)=0$

then exists $c \in (a,b)$ s.t.

$$f'(c) = 0$$

pf

Step 1 (Prove Rolle's Thm).

Suppose $g(a) = g(b) = 0$,

g is differentiable on (a,b)

and continuous on $[a,b]$.

Recall (Extreme Value Thm)

If g is continuous on $[a,b]$,

then g takes a maximum

value M and a minimum

value m on $[a,b]$.

Applying Extreme Value Thm to g.

one can find $c_0, c_1 \in [a,b]$ s.t. g takes a minimum value m at c_0

and takes a maximum value M at c_1.

<u>case1:</u> $c_0, c_1 = a$ or b

$\Rightarrow \quad M = m = g(a) = g(b) = 0$

$\Rightarrow \quad g(x) = 0 \quad \forall x \in [a, b]$

$\Rightarrow \quad g'(x) = 0 \quad \forall x \in (a, b)$

Take any $c \in (a,b)$ (eg. $c = \frac{a+b}{2}$)

we have

$$g'(c) = 0$$

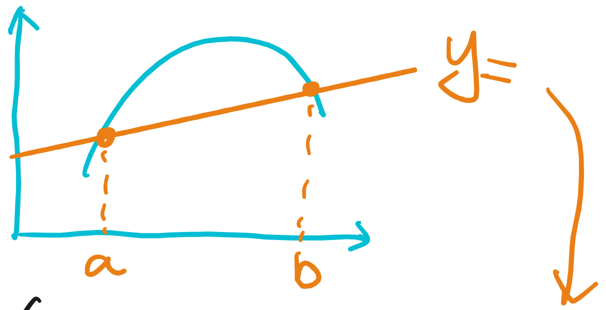
<u>case2:</u> One of c_0 and $c_1 \neq a, b$

Assume $c_0 \neq a, b. \Rightarrow c_0 \in (a,b)$

By Lemma,

$$\underline{g'(c_0) = 0}$$

$\Rightarrow$ The proof of Rolle's Thm is Complete



<u>Step2</u> (MVT)

Let

$$g(x) = f(x) - \underbrace{\left(\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right)}$$

Then

g is differentiable on (a,b) and continuous on $[a,b]$

$$g(a) = f(a) - f(a) = 0$$

$$g(b) = f(b) - \left(f(b) - \cancel{f(a)} + \cancel{f(a)}\right) = 0$$

By Step1, $\exists$ $c \in (a,b)$ s.t.

$$g'(c) = 0$$
$$\|$$
$$f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

#

Example

Let
$$f(x) = \sqrt{1-x}, \qquad x \in [-1,1]$$

By MVT, $\exists$ $c \in (-1,1)$ s.t.

$$f'(c) = \frac{f(1) - f(-1)}{1-(-1)} = -\frac{\sqrt{2}}{2}$$

Find such a c :

$$f'(x) = \left((1-x)^{\frac{1}{2}}\right)' = \frac{1}{2}\cdot(1-x)^{\frac{1}{2}-1}\cdot(1-x)'$$

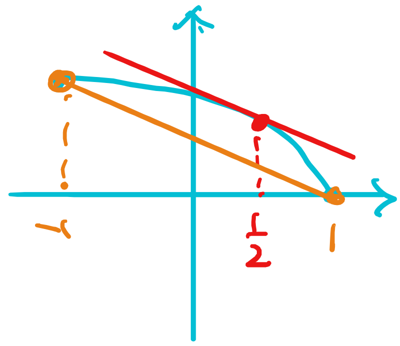
$$f'(x) = \left((1-x)^{\frac{1}{2}}\right)' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(1-x)'$$

$$= \frac{1}{2} \frac{-1}{\sqrt{1-x}}$$

Solve $\cancel{\frac{1}{2}} \dfrac{1}{\sqrt{1-c}} = \dfrac{\sqrt{2}}{\cancel{2}}$

$$\Rightarrow \sqrt{1-c} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow c = \frac{1}{2} \in (-1, 1)$$



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<u>Example</u>

Suppose f is differentiable on $(-\infty, \infty)$

Assume $f(1) = 2$ and

$$2 \le f'(x) \le 3$$

Show

$$8 \le f(4) \le 11$$

pf $([a,b]=[1,4])$

By MVT, $\exists\ c\in(1,4)$. s.t.

$$f'(c)=\frac{f(4)-f(1)}{4-1}$$

$$2\le f'(c)=\frac{f(4)-2}{3}\le 3$$

$$\Rightarrow\ 6\le f(4)-2\le 9$$

$$\Rightarrow\ 8\le f(4)\le 11 \quad \#$$