Calculus 10/17

§ Chain rule

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend left, "f \circ g"] & g(x) \arrow[r, "f"] & f(g(x)) \end{tikzcd}$$

<u>Thm</u> (Thm 3.5.6)

If g is differentiable at x and
f is differentiable at $g(x)$,

then $f \circ g$ is differentiable at x,

and $$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

Equivalently, let $y = g(x)$, $z = (f \circ g)(x)$

$$\Rightarrow \quad \left.\frac{dz}{dx}\right|_{x} = \left(\left.\frac{dz}{dy}\right|_{g(x)}\right) \cdot \left(\left.\frac{dy}{dx}\right|_{x}\right).$$

<u>Example</u>

① $\left((x^2-1)^{100}\right)' = ?$

Note:

$$g(x) = x^2 - 1 \qquad \Rightarrow \qquad (f \circ g)(x) = (x^2-1)^{100}$$
$$f(x) = x^{100}$$

Thus,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$f'(x) = (x^{100})' = 100 \cdot x^{100-1} = 100 \cdot x^{99}$$

$$g'(x) = (x^2-1)' = 2 \cdot x^{2-1} = 2x$$

$$\Rightarrow (f \circ g)'(x) = 100 \cdot (x^2-1)^{99} \cdot 2x \quad \#$$

② $$\left(\left(x + \frac{1}{x}\right)^{-3}\right)' = ?$$

Let

$$g(x) = x + \frac{1}{x} \qquad \Rightarrow \qquad (f \circ g)(x) = \left(x + \frac{1}{x}\right)^{-3}$$
$$f(x) = x^{-3}$$

So

$$(f \circ g)'(x) = \left(\left(x + \frac{1}{x}\right)^{-3}\right)'$$

$$f'(g(x)) \cdot g'(x)$$

$$= f'(g(x)) \cdot g'(x)$$

$$g'(x) = \left(x + \frac{1}{x}\right)' = (x)' + (x^{-1})'$$

$$= 1 + (-1) \cdot x^{-2} = 1 - \frac{1}{x^2}$$

$$f'(x) = (x^{-3})' = (-3) \cdot x^{-4}$$

$$\Rightarrow \left(\left(x + \frac{1}{x}\right)^{-3}\right)' = -3\left(x + \frac{1}{x}\right)^{-4} \cdot \left(1 - \frac{1}{x^2}\right)$$ #

③ $\left(\left((x+1)^5 + 1\right)^4\right)' = ?$

Let $\begin{cases} y = x+1 \\ z = y^5 + 1 \\ u = z^4 \end{cases}$ $\Rightarrow u = z^4 = (y^5+1)^4$

$$= \left((x+1)^5 + 1\right)^4$$

$$\Rightarrow \left(\left((x+1)^5 + 1\right)^4\right)' = \frac{du}{dx}$$

$$= \frac{du}{dz} \cdot \boxed{\frac{dz}{dx}} = \frac{du}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$\left(\frac{dz^4}{}\right) \quad \frac{d(y^5+1)}{} \quad \boxed{\frac{d(x+1)}{}}$$

$$= \underbrace{\frac{\quad}{dz}}_{\substack{4\cdot z^3 \\ = 4(y^5+1)^3}} \cdot \underbrace{\frac{\quad}{dy}}_{= 5\cdot y^4} \cdot \underbrace{\boxed{\frac{\quad}{dx}}}_{= 1}$$

$$= 20\cdot \left((x+1)^5+1\right)^3 \cdot (x+1)^4 \qquad \#$$

④ Let

$$y = 3u+1 \Rightarrow \frac{dy}{du} = 3$$

$$u = x^{-2} \Rightarrow \frac{du}{dx} = -2\cdot x^{-3}$$

$$x = 1-t \Rightarrow \frac{dx}{dt} = -1$$

$$\frac{dy}{dt} = ?$$

$$\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dt} = \frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{dt}$$

$$= 3\cdot(-2\cdot x^{-3})\cdot(-1)$$

$$= 6\cdot(1-t)^{-3} \qquad \#$$

§ Derivatives of trigonometric functions

<u>Thm</u> (Eq. (3.6.1), (3.6.2))

The functions

$$\sin x \quad \text{and} \quad \cos x$$

are differentiable at any $x \in \mathbb{R}$.

And,

$$\boxed{\begin{aligned} (\sin x)' &= \cos x \\ (\cos x)' &= -\sin x \end{aligned}}$$

<u>pf</u>

<u>Recall:</u> $\sin x \cdot \cos h + \cos x \cdot \sin h$

$$\text{〃}$$

$$(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \boxed{\frac{\sin x \cos h - \sin x}{h}} + \frac{\cos x \cdot \sin h}{h}$$

$$= \lim_{h \to 0} \sin x \cdot \underbrace{\frac{\cos h - 1}{h}}_{\to 0} + \cos x \cdot \underbrace{\frac{\sin h}{h}}_{\to 1}$$

Recall

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1$$

$$= \cos x = (\sin x)'$$

Similarly,

Recall: $\cos x \cos h - \sin x \sin h = \cos(x+h)$

$$(\cos x)' = \lim_{h \to 0} \frac{\boxed{\cos(x+h)} - \cos x}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{\cos x (\cos h - 1)}{h}}_{\to 0} - \underbrace{\frac{\sin x \sin h}{h}}_{\to 1}$$

$$= (\cos x)\cdot 0 - (\sin x)\cdot 1$$

$$= -\sin x = (\cos x)' \quad \#$$

$$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$$

Cor

- $$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$\cos x = \quad -\sin x \quad \shortparallel$$

$$= \frac{(\sin x)'\cdot \cos x - \sin x \cdot \boxed{(\cos x)'}}{\cos^2 x}$$

$$\shortparallel \; 1$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x = (\tan x)'$$

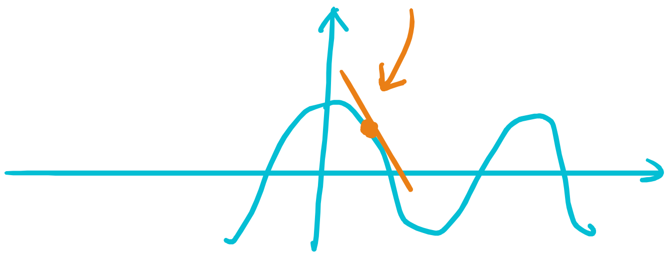
- $(\cot x)' = -\csc^2 x$
- $(\sec x)' = \sec x \cdot \tan x$
- $(\csc x)' = -\csc x \cdot \cot x$

Example

Find the tangent line of $\left(y-\frac{1}{2}\right)=\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}}\cdot\left(x-\frac{\pi}{3}\right)$

$$y = \cos x$$

at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.



<u>sol</u>

$$\frac{dy}{dx} = (\cos x)' = -\sin x$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\frac{\pi}{3}} = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$\Rightarrow$ The tangent line is

$$\left(y-\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) \quad \#$$

<u>Example</u>

① $(\cos(2x))' = ?$

Let $y = 2x$

$$\Rightarrow (\cos 2x)' = \frac{d\cos y}{d\ldots}$$

$(2x)' = 2$

$(\cos 2x)$ $\quad dx \quad (2x)' = 2$

$$\overset{\text{chain rule}}{=} \underbrace{\frac{d\cos y}{dy}}_{= -\sin y} \cdot \frac{dy}{dx}$$

$$= (-\sin y) \cdot 2$$

$$= -2\sin(2x) \qquad \#$$

② $\left(\sec(x^2+1)\right)' = ?$

Let $y = x^2 + 1 \Rightarrow \frac{dy}{dx} = 2x$

$$\Rightarrow \left(\sec(x^2+1)\right)' = \frac{d\sec y}{dx}$$

$$\overset{\text{chain rule}}{=} \frac{d\sec y}{dy} \cdot \frac{dy}{dx}$$

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

$$= \frac{d}{dy}\left(\frac{1}{\cos y}\right) \cdot (2x)$$

$$= \frac{-(-\sin y)}{\cos^2 y} \cdot 2x$$

$y = x^2+1$

$$= 2x \cdot \frac{\sin(x^2+1)}{\cos^2(x^2+1)} \quad \#$$

③ $(\sin x^\circ)' = ?$

$x^\circ = \dfrac{2\pi}{360} \cdot x$ (弧度)

$$= \left(\sin\left(\frac{2\pi}{360} \cdot x\right)\right)'$$

$y = \dfrac{2\pi}{360} x$

$\dfrac{dy}{dx} = \dfrac{\pi}{180}$

$$= \frac{d\sin y}{dy} \cdot \frac{dy}{dx}$$

x°
$\parallel$

$$= \cos y \cdot \frac{\pi}{180} = \frac{\pi}{180} \cdot \cos\left(\frac{\pi}{180} x\right)$$

$$= \frac{\pi}{180} \cdot \cos x^\circ \quad \#$$

HW4.5. 4

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

(a) $$\frac{d^n}{dx^n}(p(x)) = ? \qquad \frac{d^{n-1}}{dx^{n-1}}(p(x)) = ?$$

sol

$$\frac{d}{dx}(p) = n \cdot a_n x^{n-1} + (n-1) \cdot a_{n-1} x^{n-2} + \cdots$$

$$+ a_1 + 0$$

$$\frac{d^2}{dx^2}(p) = n(n-1) a_n x^{n-2} + (n-1)(n-2) a_{n-1} x^{n-3}$$

$$+ \cdots + 2 \cdot a_2$$

$\vdots$

$$\frac{d^{n-1}}{dx^{n-1}}(p) = n(n-1)(n-2)\cdots(2)\, a_n \cdot x$$

$$+ (n-1)(n-2) \cdots 1 \cdot a_{n-1}$$

$$\left(\frac{d^{n-1}}{dx^{n-1}} a_n x^n = n \cdot \frac{d^{n-2}}{dx^{n-2}} a_n x^{n-1} = \cdots \right.$$

$$\left. = n(n-1)\cdots 2 \quad a_n x^{\cancel{n-(n-1)}=1} \right.$$

$$= n! \cdot a_n x + (n-1)! \, a_{n-1} \quad \#$$

$$\frac{d^n}{dx^n}(p) = n! \cdot a_n \quad \#$$

(b)

$$\Rightarrow \frac{d^{n+1}}{dx^{n+1}} p = \frac{d}{dx}(n! a_n) = 0$$

$$\underset{k>n}{=} \frac{d^k}{dx^k}(p) = 0$$

HW4.5, 5

(a) $$(f \cdot g)'' = ((f \cdot g)')'$$

$$= (f' \cdot g + f \cdot g')'$$

$$= (f' \cdot g)' + (f \cdot g')'$$

$$= f'' \cdot g + f' \cdot g' + f' \cdot g' + f \cdot g''$$

$$= f''\cdot g + 2f'\cdot g' + f\cdot g'' \quad \#$$

(b) $$\left(\frac{f}{g}\right)'' = \left(\left(\frac{f}{g}\right)'\right)'$$

$$= \left(\frac{f'\cdot g - f\cdot g'}{g^2}\right)'$$

$(g\cdot g)' = g'\cdot g + g\cdot g'$

$= 2g\cdot g' \quad \|$

$$= \frac{(f'\cdot g - f\cdot g')'\cdot g^2 - (f'g - f\cdot g')\cdot (g^2)'}{(g^2)^2}$$

$$= \frac{\boxed{(f'\cdot g - f\cdot g')'}\cdot g^{\cancel{2}} - (f'g - f\cdot g')\cdot 2\cancel{g}g'}{g^{\cancel{4}\;3}}$$

$\|$

$$(f'\cdot g)' - (f\cdot g')'$$

$$= f''\cdot g + \cancel{f'\cdot g'} - \cancel{f'\cdot g'} - f\cdot g''$$

$$= f''\cdot g - f\cdot g''$$

$$= \frac{(f''\cdot g - f\cdot g'')\cdot g - 2f'\cdot g'\cdot g - 2f\cdot(g')}{g^3}$$

$$= \frac{f''\cdot g^2 - f\cdot g''\cdot g - 2f'\cdot g\cdot g' - 2f\cdot(g')^2}{g^3} \quad \#$$

HW4.2

Suppose f is continuous on $[0,1]$ and

$$0 \le f(x) \le 1 \qquad \forall\, x \in [0,1]$$

Show that $\exists\, c \in [0,1]$ s.t.

$$f(c) = c$$

pf

1. Let $g(x) = x - f(x) = h(x) - f(x)$

Since $f(x)$ and $h(x) = x$

are continuous on $[0,1]$,

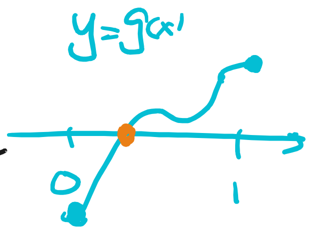
g is also continuous on $[0,1]$

2. Note that

$$g(0) = 0 - \underbrace{f(0)}_{\geq 0} \leq 0$$

$$g(1) = 1 - \underbrace{f(x)}_{\leq 1} \geq 0$$

By the intermediate value theorem, there exists $c \in [0,1]$ such that



$$g(c) = 0$$

$$\parallel$$

$$c - f(c) = 0$$

$$\Leftrightarrow \quad f(c) = c \qquad \#$$

HW1.2

Suppose

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M,$$

$$\lim_{x\to c} h(x) = N$$

Show

$$\lim_{x \to c} \left(3f(x) + 4g(x) - 5h(x)\right) = 3L + 4M - 5N$$

pf

Given $\varepsilon > 0$, by assumption

$\exists\ \delta_1, \delta_2, \delta_3 > 0$ s.t.

$|f(x) - L| < \varepsilon/12 \quad \forall\ 0 < |x - c| < \delta_1$

$|g(x) - M| < \varepsilon/12 \quad \forall\ 0 < |x - c| < \delta_2$

$|h(x) - N| < \varepsilon/12 \quad \forall\ 0 < |x - c| < \delta_3$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ s.t.

$\forall\ 0 < |x - c| < \delta$,

$$\left| \left(3f(x) + 4g(x) - 5h(x)\right) - \left(3L + 4M - 5N\right) \right|$$

$$= \left| 3(f(x) - L) + 4(g(x) - M) - 5(h(x) - N) \right|$$

$|a+b| \leq |a|+|b|$

$$\leq 3\,|f(x)-L| + 4\,|g(x)-M| + |-5| \cdot |h(x)-N|$$

$$< 3\varepsilon/12 + 4\varepsilon/12 + 5\varepsilon/12 = \varepsilon$$

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