Calculus 10/12

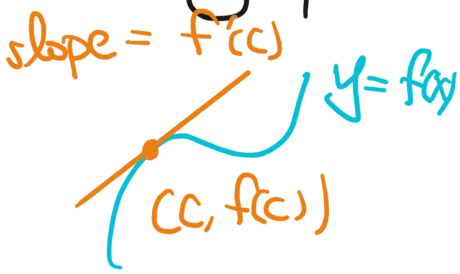
Recall

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

= slope of the tangent line of "$y = f(x)$" at $(x, f(x))$

Remark

If f is differentiable at $x = c$, then the tangent line of the graph of f at $(c, f(c))$ is



$$\boxed{y - f(c) = f'(c) \cdot (x - c)}$$

"點斜式"

Example

$f(x) = \sqrt{x}$, $\quad c = 4$

$\Rightarrow f'(c) = f'(4)$

$$= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$$

$4+h-4=h$

$/\!/ \ (a-b)(a+b) = a^2-b^2$

$$= \lim_{h\to 0} \frac{(\sqrt{4+h}-\sqrt{4})\cdot(\sqrt{4+h}+\sqrt{4})}{h\cdot(\sqrt{4+h}+\sqrt{4})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+\sqrt{4})}$$

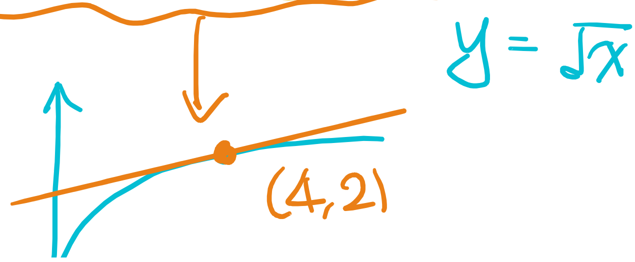
$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+\sqrt{4}} = \frac{1}{4}$$

$\rightarrow \sqrt{4}+\sqrt{4} = 4$

$\Rightarrow$ The tangent line of "$y=\sqrt{x}$"

at $(4, \sqrt{4}) = (4, 2)$ is

$$(y-2) = \overbrace{\frac{1}{4}}^{=f'(4)}\cdot(x-4) \quad \#$$



Example

① $f(x) = \sqrt[3]{x}$, $f'(0) = ?$

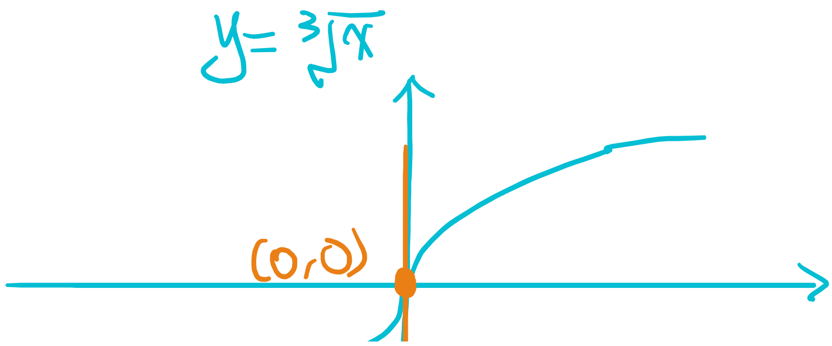
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}^{\frac{1}{3}}}{\cancel{h}^{\frac{2}{3}}}$$

$$= \lim_{h \to 0} \frac{1}{\underbrace{\sqrt[3]{h^2}}_{\to 0}}$$ does NOT exist

That is, $f(x) = \sqrt[3]{x}$ is NOT differentiable at $x = 0$ #



"slope $= +\infty$"

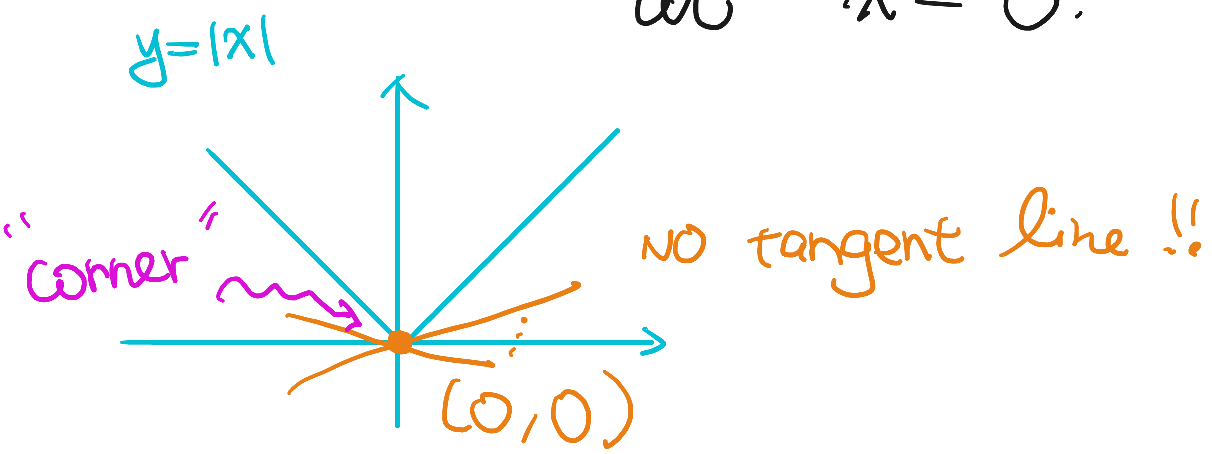
② $g(x) = |x|$, $x = 0$

$$g'(0) = \lim_{h \to 0} \frac{\overset{|0+h|}{\overset{\parallel}{g(0+h)}} - \overset{0}{\overset{\parallel}{g(0)}}}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} \frac{h}{h} = 1, & h > 0 \\ \frac{-h}{h} = -1, & h < 0 \end{cases}$$

does NOT exist.

That is, g is NOT differentiable at $x = 0$.



<u>Thm</u> (Thm 3.1.3)

If f' is differentiable at x, then f is continuous at x.

§ Differentiation rules

Thm (§3.2)

Let $f, g, f_1, \dots, f_n$ be differentiable at x

$\alpha, \alpha_1, \dots, \alpha_n$ be numbers.

Then $x \mapsto f(x)+g(x)$

(i) $(f+g)'(x) = f'(x) + g'(x)$

(ii) $(f-g)'(x) = f'(x) - g'(x)$

(iii) $(\alpha \cdot f)'(x) = \alpha \cdot f'(x)$

eg. $f(x) = \sqrt{x}$

$f'(4) = \frac{1}{4}$

$\Rightarrow (2\sqrt{x})' \Big|_{x=4} = \frac{1}{2}$

(iv) $(\alpha_1 \cdot f_1 + \alpha_2 \cdot f_2 + \cdots + \alpha_n \cdot f_n)'(x)$

$= \alpha_1 \cdot f_1'(x) + \alpha_2 \cdot f_2'(x) + \cdots + \alpha_n \cdot f_n'(x)$

(v) $$\boxed{(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)}$$

(vi) If $g(x) \neq 0$, then

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{(g(x))^2}$$

(vii) If $g(x) \neq 0$, then

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}}$$

pf

$- f(x)\cdot g(x+h) + f(x)\cdot g(x+h)$

(v)

$$(f\cdot g)'(x) = \lim_{h\to 0} \frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)\cdot \boxed{g(x+h)} - f(x)\cdot \boxed{g(x+h)}}{h} + \frac{\boxed{f(x)}\cdot g(x+h) - \boxed{f(x)}\cdot g(x)}{h}\right)$$

$\to f'(x)$ $\qquad\qquad \to g'(x)$

$$= \lim_{h\to 0}\left(\boxed{\frac{f(x+h)-f(x)}{h}}\cdot g(x+h) + f(x)\cdot\boxed{\frac{g(x+h)-g(x)}{h}}\right)$$

$g(x)$, because g is differentiable at $x \Rightarrow g$ is continuous at x

$$= f'(x)\cdot g(x) + f(x)\cdot g'(x) \qquad \#$$

(vi)

$$\left(\frac{1}{g}\right)'(x) = \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{g(x)-g(x+h)}{h\cdot \boxed{g(x+h)\cdot g(x)}}$$

$\rightarrow g(x)\cdot g(x) = (g(x))^2$

$$-\frac{g(x+h)-g(x)}{h} \rightarrow -g'(x)$$

$$= \frac{-g'(x)}{(g(x))^2} \qquad \#$$

(vii) $\left(\frac{f}{g}\right)'(x) = \left(f\cdot\frac{1}{g}\right)'(x)$

$$\overset{(v)}{=} f'(x)\cdot \frac{1}{g(x)} + f(x)\cdot \left(\frac{1}{g}\right)'(x)$$

$$\overset{(vi)}{=} \frac{f'(x)}{g(x)} + f(x)\cdot\left(\frac{-g'(x)}{(g(x))^2}\right)$$

$$= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

#

Cor (Derivatives of polynomials, Equation (3.2.8))

(viii) If $f(x) = \alpha =$ constant function, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\overbrace{\alpha - \alpha}^{=0}}{h} = 0$$

(ix) If $f(x) = x$, then

$$f'(x) = \lim \frac{f(x+h)-f(x)}{}$$

$$= \lim_{h\to 0} \frac{\overbrace{(x+h) - x}^{= \cancel{h}}}{\cancel{h}} = 1$$

i.e. $$(x)' = 1$$

(x) $f(x) = x^n$, $f'(x) = n \cdot x^{n-1}$

pf

$n=2$: $$(x^2)' = (x \cdot x)'$$

$$\overset{(v)}{=} \underbrace{(x)'}_{=1} \cdot x + x \cdot \underbrace{(x)'}_{=1}$$

$$\overset{(ix)}{=} 1 \cdot x + x \cdot 1 = 2x$$

$n=3$: $$(x^3)' = (x \cdot x^2)'$$

$$\overset{(v)}{=} \overbrace{(x)'}^{1 =} \cdot x^2 + x \cdot \overbrace{(x^2)'}^{2x =}$$

$$= 3x^2$$

$\vdots$

$$(x^n)' = (x \cdot x^{n-1})'$$

$$(x^n)' = (x \cdot x \quad)' = \cdots$$

$$= n \cdot x^{n-1} \qquad \#$$

(xi) $\left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\right)'$

$$\overset{(iv)}{=} a_n \cdot (x^n)' + a_{n-1} (x^{n-1})' + \cdots + (a_0)'$$

$$\overset{(x)}{=} n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

<u>Example</u>

① $\left((x^2-1)\, x^3 (x+1)\right)'$

$$\overset{(v)}{=} (x^2-1)' \cdot (x^3(x+1)) + (x^2-1) \cdot \underbrace{(x^3 \cdot (x+1))'}_{\overset{(v)}{=}}$$

$$(x^3)' \cdot (x+1) + x^3 \cdot (x+1)'$$

$$= \underbrace{(x^2-1)'}_{\overset{2x}{\overset{\|(xi)}{}}} \cdot x^3 \cdot (x+1) + (x^2-1) \cdot \underbrace{(x^3)'}_{\overset{3x^2}{\|}} \cdot (x+1)$$

$$+ (x^2-1) \cdot x^3 \cdot (x+1)'$$

$$+ (x^2-1)\cdot x^3\cdot \underbrace{(x+1)'}_{=1}$$

$$= 2x\cdot x^3\cdot(x+1) + (x^2-1)\cdot 3x^2\cdot(x+1)$$
$$+ (x^2-1)\cdot x^3\cdot 1 \qquad \#$$

② $$\left(\frac{1}{x^n}\right)' = \left(x^{-n}\right)' \qquad \leftarrow \left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

$$\overset{(vi)}{=} \frac{-(x^n)'}{(x^n)^2}$$

$$\overset{(x)}{=} \frac{-n\cdot \cancel{x^{n-1}}}{\cancel{x^{2n}}} \qquad 2n-(n-1)=n+1 \qquad = -n\cdot\frac{1}{x^{n+1}}$$

$$= \boxed{\begin{array}{c}(-n)\cdot x^{-n-1}\\ \parallel \\ (x^{-n})'\end{array}} \qquad \#$$

③ $$\left(\frac{6x^2-1}{\quad}\right)'$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

(vii)

$$= \frac{\boxed{(6x^2-1)'}_{=12x} \cdot (x^4+5x+1) - (6x^2-1) \cdot \boxed{(x^4+5x+1)'}_{4x^3+5}}{(x^4+5x+1)^2}$$

$$= \frac{12x^5 + 60x^2 + 12x - 24x^5 - 30x^2 + 4x^3 + 5}{(x^4+5x+1)^2}$$

$$= \frac{-12x^5 + 4x^3 + 30x^2 + 12x + 5}{(x^4+5x+1)^2}$$

#

④ Find the tangent line of the graph of $f(x) = \frac{3x}{1-2x}$ at $(2, -2)$.

sol

(vii)

$$f'(x) = \frac{\underbrace{(3x)'}_{3} \cdot (1-2x) - (3x) \cdot \underbrace{(1-2x)'}_{-2}}{(1-2x)^2}$$

$$= \frac{3 - 6x + 6x}{(1-2x)^2} = \frac{3}{(1-2x)^2}$$

$\Rightarrow f'(2) = \dfrac{3}{(1-2\cdot 2)^2} = \dfrac{3}{9} = \dfrac{1}{3}$

$\Rightarrow$ The tangent line is

$$y+2 = \left(y-(-2)\right) = \frac{1}{3}\cdot(x-2) \quad \#$$

⑤ Find the point(s) on the graph of

$$f(x) = \frac{4x}{x^2+4}$$

where the tangent line is horizontal (i.e. slope $=0$)

sol

Solve

$$f'(x) = 0$$

$$\|$$

$$\left(\frac{4x}{x^2+4}\right)' \overset{\text{(v.iii)}}{=} \frac{\overset{4}{\overset{\|}{(4x)'}}(x^2+4)-(4x)\overset{2x}{\overset{\|}{(x^2+4)'}}}{(x^2+4)^2}$$

$$= \frac{4x^2+16-8x^2}{(x^2+4)^2}$$

$$= \frac{16-4x^2}{\underbrace{(x^2+4)^2}_{\neq 0}} = 0$$

$$\Leftrightarrow \quad 16-4x^2=0$$

$$\Leftrightarrow \quad x^2 = \frac{16}{4} = 4$$

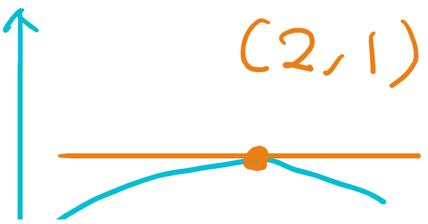
$$\Leftrightarrow \quad x = \pm 2$$

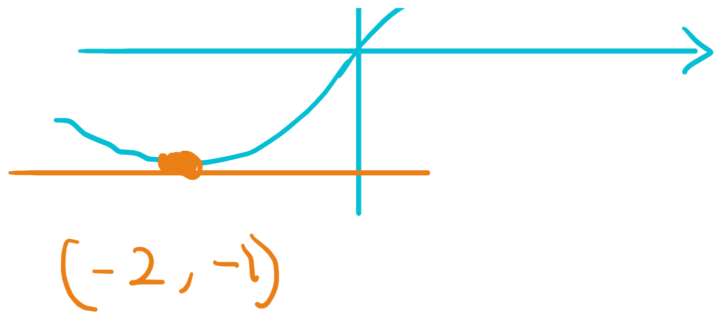
So the tangent lines at

$$\left(+2,\ \underset{1}{\underset{\parallel}{f(2)}}\right) \text{ and } \left(-2,\ \underset{-1}{\underset{\parallel}{f(-2)}}\right)$$

are horizontal.

$$y = f(x) = \frac{4x}{x^2+4}$$





<u>Def</u> (Higher derivatives, §3.3)

$$f'(x) = \frac{df}{dx}$$

$$f''(x) = \frac{d^2 f}{dx^2} = \text{sencond order derivative of } f$$

$$= (f')' = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

e.g. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$$\Rightarrow f''(x) = (3x^2)' = 2 \cdot 3 \cdot x = 6x$$

$$f^{(3)}(x) = f'''(x) = (f''(x))'$$

$$= (6x)' = 6$$

The n-th order derivative of f is

denoted by

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

<u>Example</u>

<u>Recall</u>

$$(x^{-n})' = -n \cdot x^{-n-1}$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$\Rightarrow$

$$f'(x) = \frac{df}{dx} \overset{(vi)}{=} \frac{-1}{x^2} = \boxed{-x^{-2}}$$

$$f''(x) = \frac{d^2 f}{dx^2} = \left(\frac{-1}{x^2}\right)' = -\frac{-(x^2)'}{(x^2)^2}$$

$$= 2x^{-3} = \frac{2}{x^3}$$

$$f'''(x) = \frac{d^3 f}{dx^3} = (2x^{-3})' = 2 \cdot (-3) \cdot x^{-3-1}$$

$$= -6x^{-4} = -\frac{6}{x^4}$$

$\vdots$

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = (-1)^n \cdot n! \cdot x^{-n-1}$$

#