Calculus 10/3

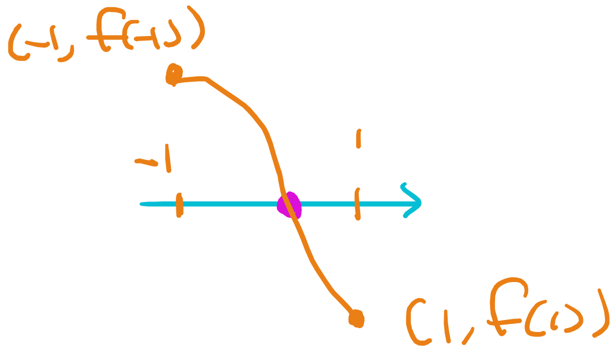
<u>Example</u>

Show that

$$x^4 - x - 1 = 0 \quad (*)$$

has a solution in $[-1, 1]$.

pf



The function

$$f(x) = x^4 - x - 1$$

is continuous on $[-1, 1]$.

Since

$$f(-1) = (-1)^4 - (-1) - 1 = 1 > 0$$

$$f(1) = 1^4 - 1 - 1 = -1 < 0,$$

by Intermediate Value Thm,

$\exists\ c \in (-1, 1)$ s.t.

$$f(c) = 0$$

That is, c is a solution of $(*)$ $\#$

<u>Example</u>

Show that

$$x^3 - 4x + 2 = 0 \quad (**)$$

has 3 distinct roots in $[-3, 2]$.

<u>pf</u>

The function

$$g(x) = x^3 - 4x + 2$$

is continuous on $[-3, 2]$.

Since

$$g(-3) = -27 + 12 + 2 = -13 < 0$$

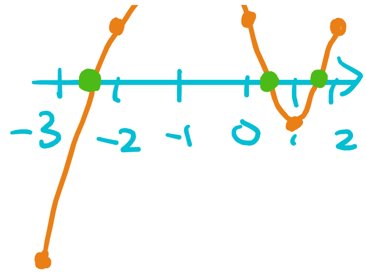
$$g(-2) = 2 > 0$$

$$g(-1) = -1 + 4 + 2 = 5 > 0$$

$$g(0) = 2 > 0$$

$g(1) = -1 < 0$

$g(2) = 2 > 0$,



by Intermediate Value Thm,

$\exists\ c_1 \in (-3,-2)$, $c_2 \in (0,1)$,

$c_3 \in (1, 2)$ s.t.

$$g(c_1) = g(c_2) = g(c_3) = 0.$$

So (**) has 3 distinct roots in $[-3, 2]$ #

Application to extreme values and boundedness

Thm (Extreme Value Thm, Thm 2.6.2)

If f is a continuous function

on a closed interval $[a,b]$,

then f takes an (absolute) maximum value M on $[a,b]$

(i.e. $\exists\, c_1 \in [a,b]$ s.t.

$f(x) \le f(c_1) = M \quad \forall x \in [a,b]$)

and f takes an (absolute) minimum value m on $[a,b]$

(i.e. $\exists\, c_2 \in [a,b]$ s.t.

$f(x) \ge f(c_2) = m \quad \forall x \in [a,b]$)

In particular,

上界

下界 $m \le f(x) \le M \quad \forall x \in [a,b]$.

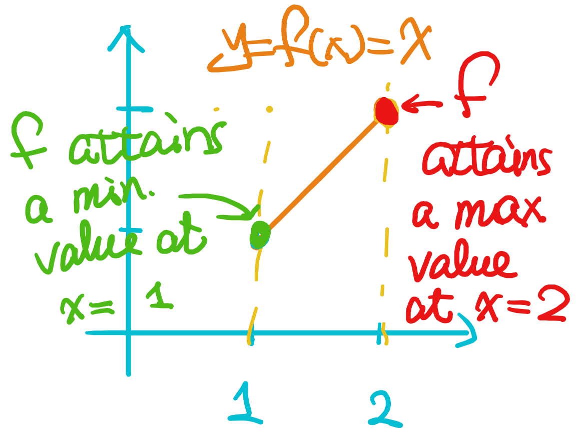
$\Rightarrow$ f is bounded 有界 on $[a,b]$

Example

① $f(x) = x$ is continuous on $[1, 2]$

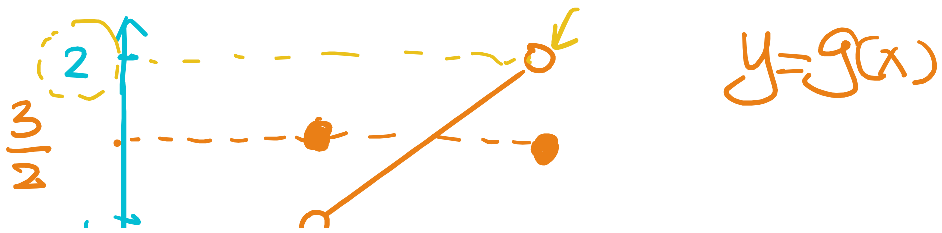
$f(x) = x$ is continuous on $[1,2]$

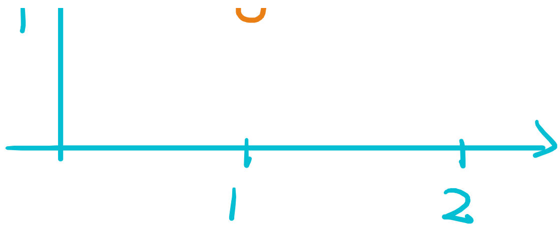
$\Rightarrow$ f takes a max. value and a min. value on $[1,2]$



② $$g(x) = \begin{cases} \frac{3}{2} & \text{if } x = 1 \\ x & x \in (1,2) \\ \frac{3}{2} & x = 2 \end{cases}$$

is NOT continuous on $[1,2]$, and does NOT attain a max. value neither a min. value on $[1,2]$.

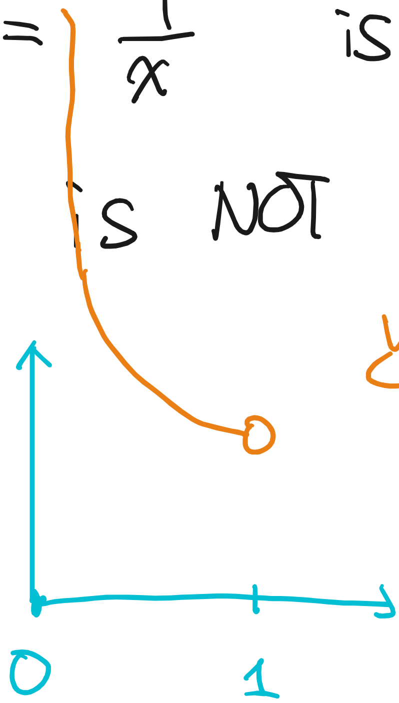




③ $h(x) = \frac{1}{x}$ is continuous $\underset{\sim\sim\sim}{(0,1)}$ NOT a closed interval

and is NOT bounded above

$y = h(x) = \frac{1}{x}$



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§3. Derivatives

Def (Def. 3.1.1)

A function f is differentiable 可微

at x if the limit

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}}$$

exists. If this limit exists, it is called the derivative of f at x, denoted by $f'(x)$ or $\left.\dfrac{df}{dx}\right|_x$

導數，微分

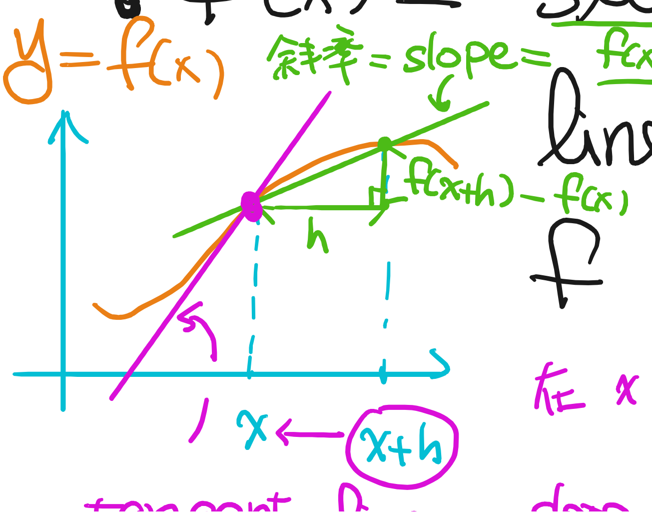
Remark

- $f'(x)$ = the rate of change of f at x

- $f'(x)$ = slope of the tangent line of the graph of f at $(x, f(x))$

斜率

斜率 = slope = $\dfrac{f(x+h)-f(x)}{h}$ = 平均變化率



在 x 的瞬間變化率

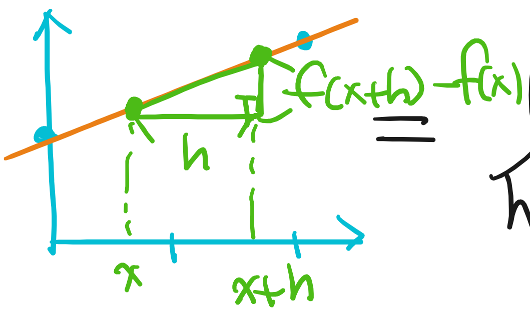
tangent line, slope $= f'(x)$

Example

① $f(x) = \frac{1}{2}x + 1$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$y = f(x) = \frac{1}{2}x + 1$



$$= \lim_{h \to 0} \frac{\left(\frac{1}{2}(\cancel{x}+h)+\cancel{1}\right) - \left(\cancel{\frac{1}{2}x} + \cancel{1}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2}\cancel{h}}{\cancel{h}} = \frac{1}{2}$$

② $g(x) = x^2$. Find $g'(1)$.

<u>sol</u>.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim \frac{(x+h)^2 - x^2}{}$$

$a^2 - b^2$

$= (a+b)(a-b)$

$$\lim_{h\to 0} \qquad h$$

$$= \lim_{h\to 0} \frac{(x+h+x)\boxed{(x+h-x)}}{\cancel{h}}$$

$\cancel{h}$ $\parallel$

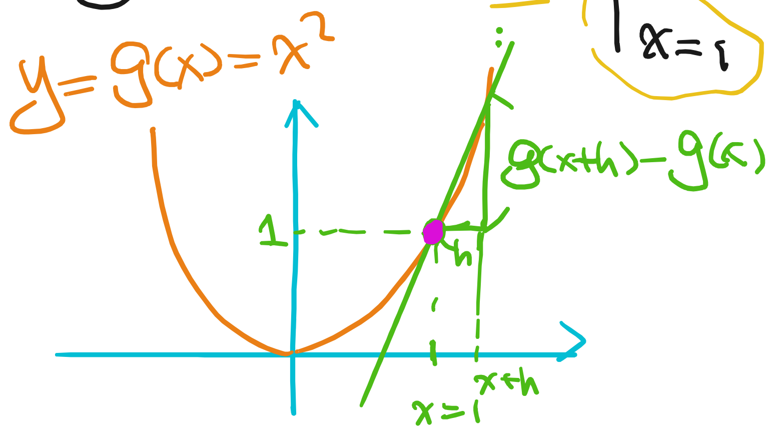
$$= \lim_{h\to 0} 2x+h = 2x$$

$x=1$ 代入

So

$$g'(1) = 2x\Big|_{x=1} = 2\cdot 1 = 2$$

\#



③ Let $\tilde{h}(x) = \sqrt{x}$.

Find the rate of change $\tilde{h}(x)$ with respect to x at $x=4$.

$\underline{\text{Sol}}$

$$\text{Ans} = \tilde{h}'(4)$$

$$= \lim_{h\to 0} \frac{\tilde{h}(4+h) - \tilde{h}(4)}{h}$$

$a^2 - b^2$
$= (a+b)(a-b)$

$$= \lim_{h\to 0} \frac{\left(\sqrt{4+h} - \sqrt{4}\right)\left(\sqrt{4+h} + \sqrt{4}\right)}{h\left(\sqrt{4+h} + \sqrt{4}\right)}$$

$$= \lim_{h\to 0} \frac{\underbrace{(4+h) - 4}_{= \cancel{h}}}{\cancel{h}\left(\sqrt{4+h} + \sqrt{4}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + \sqrt{4}} \to \sqrt{4} + \sqrt{4} = 4$$

$$= \frac{1}{4}$$

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