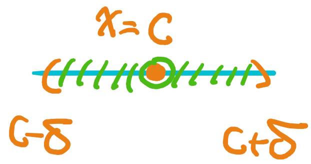
Calculus 9/26

Def

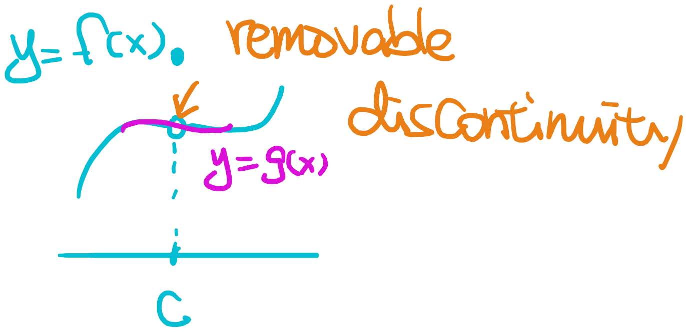
Assume f is discontinuous at $x = c$

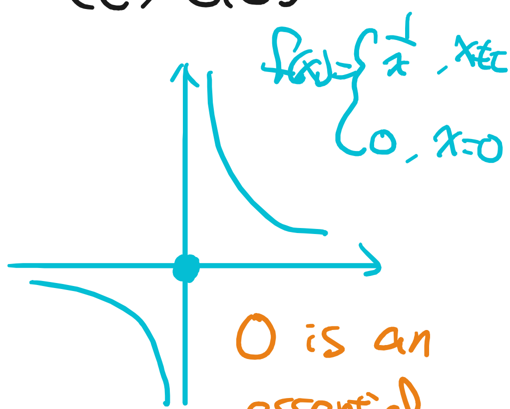
We say c is a removable discontinuity of f if $\exists\ \delta > 0\ \exists$ continuous function g on $(c-\delta, c+\delta)$



s.t.

$$f(x) = g(x) \qquad \forall\ x \in (c-\delta, c) \cup (c, c+\delta)$$





0 is an essential discontinuity

We say f has an essential discontinuity at $x = c$ if c is NOT removable.

Example

Determine the continuity of

the following functions.
Classify the discontinuities of the function if it has any.

① $f(x) = \begin{cases} 3|x| + \dfrac{x^3-x}{x^2-5x+6}, & x \neq 2,3 \\ 0 & x = 2,3 \end{cases}$

<u>sol</u>

(i) For $x \neq 2, 3$,

$$x^2 - 5x + 6 = (x-2)(x-3) \neq 0$$

$\overset{\shortparallel}{(-2)\times(-3)}$

$\Rightarrow$ $\dfrac{x^3-x}{x^2-5x+6}$ is continuous at any $x \neq 2, 3$.

Since $|x|$ is continuous,

$$3|x| + \frac{x^3-x}{x^2-5x+6}$$

is Continuous at any $x \neq 2, 3$

(ii) $x=2$:

$$\left(\overset{?}{f(2)} = 0\right)$$

$$3|2| = 6 \qquad 2^3 - 2 = 6 \neq 0$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(3|x| + \frac{x^3 - x}{x^2 - 5x + 6}\right)$$

$$2^2 - 5\cdot 2 + 6 = 0$$

lim does NOT exist !!

does NOT exist

$\begin{cases} \lim\limits_{x\to c} f(x) \text{ exists} \\ \lim\limits_{x\to c} g(x) \text{ doesn't exist} \end{cases}$ $\quad g = (f+g) - f$

$\Rightarrow \lim\limits_{x\to c}(f(x)+g(x))$ doesn't exist

$\Rightarrow$ f has an essential discontinuity at $x=2$

Remark

Assume f is discontinuous at $x=c$

c is removable $\Leftrightarrow \lim\limits_{x\to c} f(x)$ exists

c is essential $\Leftrightarrow \lim\limits_{x\to c} f(x)$ doesn't exist

(iii) $x=3$: Similar to (ii),

$$9 \qquad 3^3 - 3 = 24 \neq 0$$

$$\lim f(x) = \lim\left(3|x| + \frac{x^3 - x}{\quad}\right)$$

$$\lim_{x\to 3} f(x) = \lim_{x \to 3} \frac{5x+1}{x^2-5x+6}$$

$\underbrace{x^2-5x+6}_{\to 0}$

does NOT exist

$\Rightarrow$ f has an essential discontinuity at $x = 3$ #

② $f(x) = \sqrt{\dfrac{x^2+1}{x-3}}$ is continuous on $(3, \infty)$

because

$$f = f_1 \circ f_2$$

where

$f_1(x) = \sqrt{x}$ ← continuous on $(0, \infty)$

$f_2(x) = \dfrac{x^2+1}{x-3}$ ← continuous at any $x \neq 3$

Also note that $\underline{\quad} > 0$

$$f_2(x) = \frac{x^2+1}{x-3} > 0 \quad \forall x > 3$$

So at any $x > 3$,

$$f(x) = f_1(f_2(x)) = \sqrt{\frac{x^2+1}{x-3}}$$

is continuous #

③ $$f(x) = \frac{1}{5-\sqrt{x^2+16}}$$

$$= \frac{1}{g(x)}, \text{ where } g(x) = 5 - \sqrt{x^2+16}$$

$$\sqrt{x^2+16} = g_1(g_2(x)), \text{ where}$$

$g_1(x) = \sqrt{x}$ ← Continuous on $[0,\infty)$

$g_2(x) = x^2+16$ ← Continuous on $(-\infty,\infty)$

$> 0 \quad \forall x \in (-\infty, \infty)$

$\Rightarrow \sqrt{x^2+16} = g_1(g_2(x))$ is Continuous on $(-\infty, \infty)$

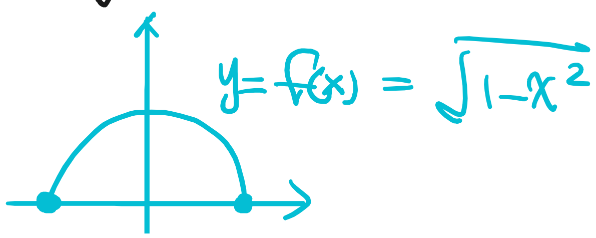
$\Rightarrow g(x) = 5 - \sqrt{x^2+16}$ is Continuous on $(-\infty, \infty)$

$\Rightarrow f(x) = \dfrac{1}{g(x)}$ is Continuous at any $x \neq \pm 3$

$g(x) = 0 = 5 - \sqrt{x^2+16} \iff x = \pm 3$

Furthurmore, $f(x)$ has essential discontinuities at $x = \pm 3$ #

④ $f(x) = \sqrt{1-x^2}$ is Continuous on $[-1, 1]$



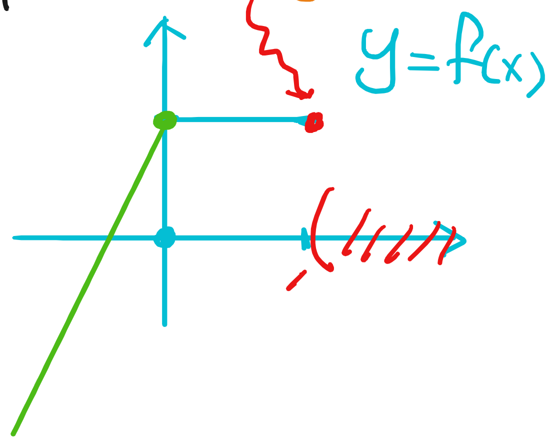
⑤ $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ is Continuous on $(-1, 1)$

$2\cdot 0+1 = 1$

⑥ $f(x) = \begin{cases} \underbrace{2x+1}_{} , & x \le 0 \\ 1 , & 0 < x \le 1 \\ \underbrace{x^2+1}_{} , & x > 1 \end{cases}$

$1^2+1=2$

essential discontinuity



is Continuous on $(-\infty, 1]$
and on $(1, \infty)$

But discontinuous on $(-\infty, \infty)$

Application to limits

Thm

If $\lim\limits_{x\to c} f(x) = b$ and g is

Continuous at $x = b$, then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right)$$

$$= g(b)$$

Example

Recall

$\sin x$, $\cos x$ are continuous

$$\lim_{x \to \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$$

$$= \cos\left(\lim_{x \to \frac{\pi}{2}} \left(2x + \sin\left(\frac{3}{2}\pi + x\right)\right)\right)$$

$$= 2 \cdot \frac{\pi}{2} = \pi$$

$$= \cos\left(\boxed{\lim_{x \to \frac{\pi}{2}} 2x} + \lim_{x \to \frac{\pi}{2}} \sin\left(\frac{3}{2}\pi + x\right)\right)$$

$$= \frac{3}{2}\pi + \frac{\pi}{2} = 2\pi$$

$$= \cos\left(\pi + \sin\left(\boxed{\lim_{x \to \frac{\pi}{2}} \left(\frac{3}{2}\pi + x\right)}\right)\right)$$

$$= \cos(\pi + \sin(2\pi)) = \cos\pi = -1$$

#

Application to "solving" equations:

中間值定理

Thm (Intermediate Value Thm, Thm 2.6.1)

Let f be a continuous function on $[a,b]$.

If there is a number k s.t. either

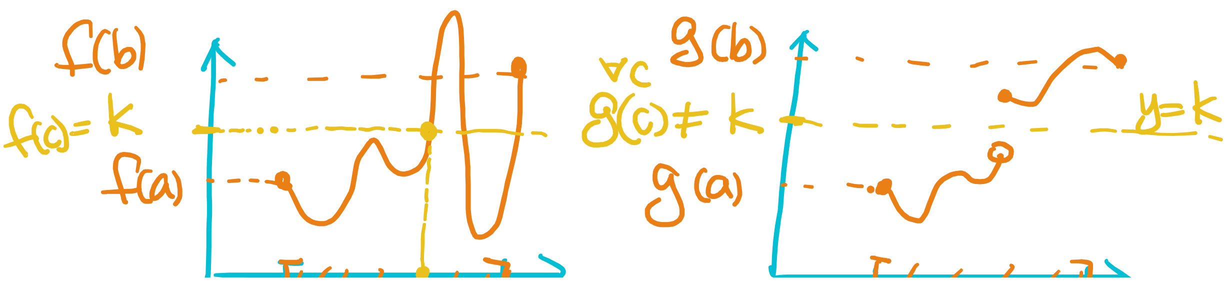
$$f(a) < k < f(b)$$

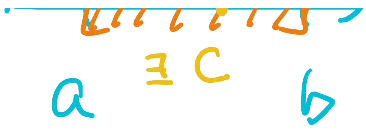
or

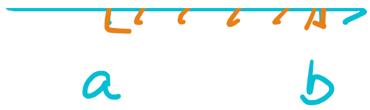
$$f(a) > k > f(b),$$

then there exists $c \in (a,b)$ s.t.

$$f(c) = k$$




a
∃c
b


a
b