Calculus 9/21

Recall

- $\lim_{x \to c} \sin x = \sin c$
- $\lim_{x \to c} \cos x = \cos c$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Cor

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$\to 1 - \cos 0 = 1 - 1 = 0$

$x \to 0$

$a = 1$

$a^2 - b^2 = (a+b)(a-b)$ $\quad b = \cos x$

$\cos^2 x + \sin^2 x = 1 \iff 1 - \cos^2 x = \sin^2 x$

Pf

Note $(x \neq 0)$

$$\frac{1 - \cos x}{x} = \frac{(1 - \cos x)(1 + \cos x)}{x\,(1 + \cos x)}$$

$$= \frac{1^2 - \cos^2 x}{x(1 + \cos x)} = \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

Since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$\to \sin 0 = 0$

$$\lim_{x\to 0} \frac{\sin x}{1+\cos x} = \frac{0}{2} = 0,$$

$\to 1+\cos 0 = 1+1 = 2 \neq 0$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 1 \cdot 0 = 0 \quad \#$$

<u>Cor</u>

Let $a \neq 0$. $y = ax$

(i) $$\lim_{x\to 0} \frac{\sin(ax)}{ax} = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

(ii) $$\lim_{x\to 0} \frac{1-\cos(ax)}{ax} = \lim_{y\to 0} \frac{1-\cos y}{y} = 0$$

<u>Example</u>

① $$\lim_{x\to 0} \frac{\sin(4x)}{3x} = \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{4}{3} = 1 \cdot \frac{4}{3} = \frac{4}{3}$$ #

② $$\lim_{x\to 0} x \cdot \cot(3x) = \lim_{x\to 0} x \cdot \frac{\cos(3x)}{\sin(3x)}$$

$$= \lim_{x\to 0} \frac{\cos(3x)}{3 \cdot \boxed{\frac{\sin(3x)}{3x}}}$$

$\cos(3 \cdot 0) = 1$

Cor $\xrightarrow[x\to 0]{\text{as}}$ $1 \neq 0$

$$= \frac{\lim_{x\to 0} \cos(3x)}{\lim_{x\to 0} 3 \cdot \frac{\sin(3x)}{3x}} = \frac{1}{3}$$ #

③ $$\lim_{x\to \frac{\pi}{4}} \frac{\sin(x-\frac{\pi}{4})}{(x-\frac{\pi}{4})^2}$$

Let $y = x - \frac{\pi}{4}$

$y \to 0 \Leftrightarrow x \to \frac{\pi}{4}$

$$= \lim_{y\to 0} \frac{\sin y}{y^2} = \lim_{y\to 0} \frac{\frac{\sin y}{y}}{y}$$

$\frac{\sin y}{y} \to 1 \neq 0$

$y \to 0$

$\Rightarrow$ The limit does NOT exist #

④ $$\lim_{x\to 0}\frac{x^2}{\sec x - 1} = \lim_{x\to 0}\frac{x^2 \times \cos x}{\left(\frac{1}{\cos x} - 1\right)\times \cos x}$$

$$= \lim_{x\to 0}\frac{x^2\cos x \cdot (1+\cos x)}{(1-\cos x)\cdot(1+\cos x)}$$

$\to 0^2 \cdot \cos 0 = 0$

$1 - \cos 0 = 0$

$$= \lim_{x\to 0}\frac{x^2\cos x\,(1+\cos x)}{1-\cos^2 x}$$

$1-\cos^2 x = \sin^2 x$

$$= \lim_{x\to 0}\frac{\cos x\cdot(1+\cos x)}{\left(\frac{\sin x}{x}\right)^2}$$

$\to \cos 0 \cdot (1+\cos 0) = 1\cdot(1+1) = 2$

$\to 1^2 = 1 \neq 0$

$$= \frac{2}{1} = 2 \quad \#$$

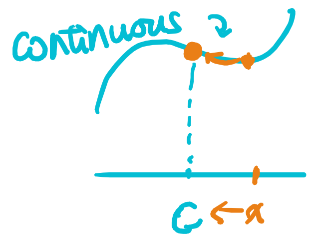
§ Continuity (連續, §2.4)

Def (Def 2.4.1 & 2.4.5)

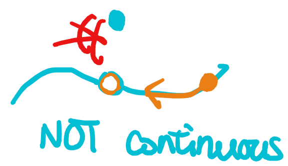
Let f be a function, c be a number

- We say f is continuous at c if f is defined in $(c-\delta, c+\delta)$ for some $\delta>0$ and

$$\lim_{x\to c} f(x) = f(c)$$

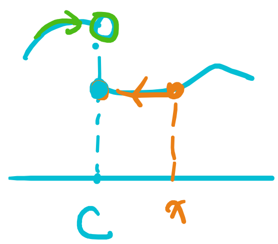


- We say f is discontinuous at c if f is NOT continuous at c



- We say f is (left-continuous) right-continuous (continuous from the left) (or continuous from the right) at c if f is defined in $[c, c+\delta)$ ($(c-\delta, c]$) for some $\delta>0$, and

$$\lim_{x\to c^+} f(x) = f(c)$$



$$\left(\lim_{x\to c^-} f(x) = f(c) \right)$$

- The function f is continuous on the open interval (a,b) if f is continuous at any $c \in (a,b)$.
- f is continuous on $[a,b]$ if
  (i) f is continuous on (a,b) and
  (ii) f is right-continuous at a
  (iii) f is left-continuous at b
- Similarly, "f is continuous" can be defined over $(a,b]$, $[a,b)$, $(-\infty,b]$, $[a,\infty)$, $(-\infty,\infty)$

Remark

f is continuous at c

$\Leftrightarrow$ (i) $f(c)$ exists

(ii)

(ii) $\lim_{x \to c} f(x)$ exists

(iii) $f(c) = \lim_{x \to c} f(x)$

Remark

The following functions are continuous on $(-\infty, \infty)$:

① any polynomials

② $\sin x$

③ $\cos x$

④ $|x|$

Note:

$\tan x$ is NOT continuous on $(-\infty, \infty)$

Moreover, $\sqrt[n]{x}$ is continuous on $[0, \infty)$

Example

Determine whether the function is continuous

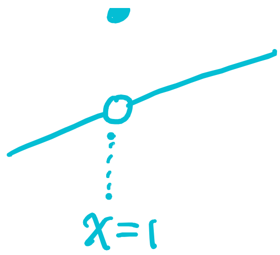
① $f(x) = \frac{1}{2}x + 1$ is continuous on $(-\infty, \infty)$

∃ continuous f s.t. $f(x) = g(x)$ $\forall x \neq 1$

"removable distinuity"

② $g(x) = \begin{cases} \frac{1}{2}x + 1, & x \neq 1 \end{cases}$

$\quad 3 \quad , \quad x=1$

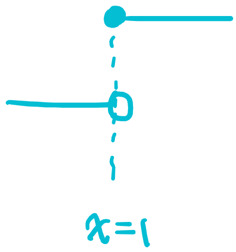


$$\lim_{x\to 1} g(x) = \lim_{x\to 1}\left(\tfrac{1}{2}x+1\right) = \frac{3}{2} \neq 3$$

So g is discontinuous at $x=1$

and continuous at $x \neq 1$.

③ $$h(x) = \begin{cases} 1 & , \ x<1 \\ 2 & , \ x \geq 1 \end{cases}$$



"jump distinuity"

$\lim_{x\to 1} h(x)$ does NOT exist.

So h is discontinuous at $x=1$

" right-continuous at $x=1$

continuous at any $x \neq 1$

<u>Thm</u> (Thm 2.4.2)

If f and g are continuous at c, then (k : number)

$f+g$, $f-g$, $f\cdot g$, $k\cdot g$

are continuous at c

$$\lim_{x\to c}(f+g) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c) = (f+g)(c)$$

Furthermore, if $g(c) \neq 0$, then

$\frac{f}{g}$ is continuous at c.

<u>Thm</u> (Thm 2.4.4)

If

(i) g is continuous at c

(ii) f is continuous at $g(c)$

then $f\circ g$ is continuous at c

e.g. $\sin^2 x$ is continuous

$f(x) = x^2$

$g(x) = \sin x$

$\Rightarrow (f \circ g)(x) = (\sin x)^2 = \sin^2 x$