Calculus 9/19

Remark

All the properties we reviewed hold for "one-side limits" and "limits involving infinity"

Example

①

$-x$ if $x<0$ $\qquad$ -1 $\qquad$ $\sqrt{0+1}=\sqrt{1}=1$

$$\lim_{x\to 0^-} \frac{|x|}{x}\cdot\frac{\sqrt{x+1}}{x-1} = \lim_{x\to 0^-} \frac{-\cancel{x}}{\cancel{x}}\cdot\frac{\sqrt{x+1}}{x-1}$$

$x<0$ $\qquad$ $0-1=-1$ $\neq 0$

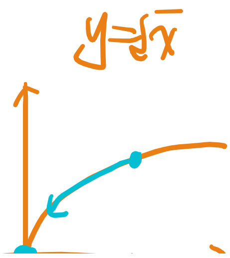
$$= (-1)\cdot\frac{\sqrt{1}}{-1} = 1 \quad \#$$

②

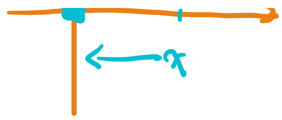
$$\lim_{x\to 0^+} \frac{x^{\frac{3}{2}}}{|x|} = \lim_{x\to 0^+} \frac{x^{\frac{3}{2}}}{x^1} = x^{\frac{3}{2}-1} = \sqrt{x}$$

$|x| = x$ $\qquad$ $x>0$

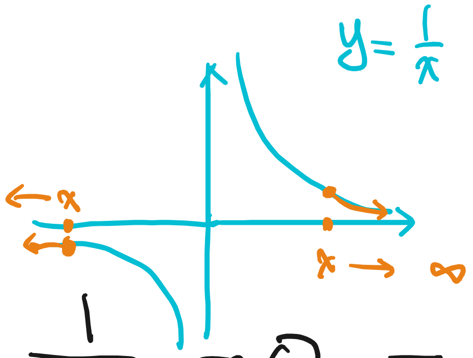
$$= \lim \sqrt{x} = \sqrt{0} = 0$$



$x \to 0^+$



Example



① $$\lim_{x \to \infty} \frac{1}{x} = 0 = \lim_{x \to -\infty} \frac{1}{x}$$

② $$\lim_{x \to \infty} \frac{1}{x^2} = \lim_{x \to \infty} \frac{1}{x} \cdot \frac{1}{x}$$

$$\overset{\text{Property}}{=} \left(\lim_{x \to \infty} \frac{1}{x} \right) \cdot \left(\lim_{x \to \infty} \frac{1}{x} \right)$$

$$\overset{①}{=} 0 \cdot 0 = 0$$

③ $$\lim_{x \to \infty} \frac{\underbrace{x^2 + x}_{\to \infty} \times \frac{1}{x^2}}{\underbrace{2x^2 + 1}_{\to \infty} \times \frac{1}{x^2}} = \lim_{x \to \infty} \frac{(\cancel{x^2} + \cancel{x}) \cdot \frac{1}{\cancel{x^2}}}{(2\cancel{x^2} + \cancel{1}) \frac{1}{\cancel{x^2}}}$$

$$= \lim_{x \to \infty} \frac{1 + \overset{\to 0}{\frac{1}{x}}}{2 + \frac{1}{x^2}}$$

$$\leftarrow \lim_{x \to \infty} \left(1 + \frac{1}{x}\right) = \lim_{x \to \infty} 1 + \lim_{x \to \infty} \frac{1}{x}$$

$$= 1 + 0 = 1$$

$$\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{1}{x^2} = 2+0 = 2 \neq 0$$

$$= \frac{1}{2} \quad \#$$

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§ Pinching thm and trigonometric limits

夾擠定理

<u>Thm</u> (Thm 2.5.1)

Suppose $\exists\, p>0$ s.t.

$$0<|x-c|<p \Rightarrow \boxed{h(x) \leq f(x) \leq g(x)}$$

Then if

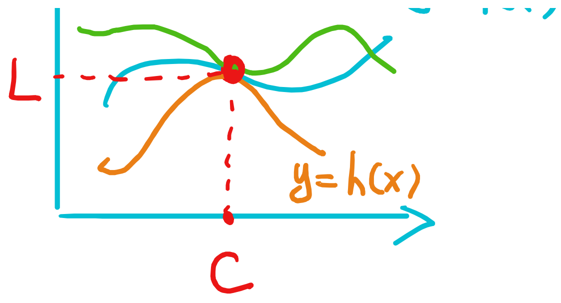
$$\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L,$$

$y=g(x)$

$y=f(x)$

then

$$\lim_{x \to c} f(x) = L$$



<u>Thm</u> (Eq. (2.5.2), Eq (2.5.3))

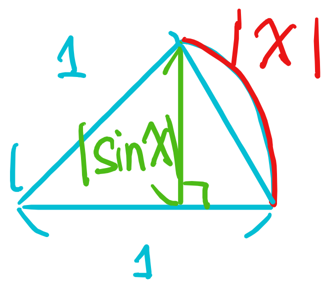
(i) $\lim_{x \to 0} \sin x = 0$

(ii) $\lim_{x \to 0} \cos x = 1$

Pf = proof

弧度 $= r \cdot |\theta|$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



For $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

$$0 \le |\sin x| \le |x|$$

Since

$$\lim_{x \to 0} 0 = \lim_{x \to 0} |x| = 0,$$

by Pinching Thm,

$$\lim_{x\to 0} |\sin x| = 0$$

$$\Leftrightarrow \lim_{x\to 0} \sin x = 0 \quad \text{— (i)}$$

Remark

$(f(x) = \sin x,\ L = 0)$

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c} |f(x) - L| = 0$$

Furthermore, $\boxed{\cos^2 x + \sin^2 x = 1}$

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \sqrt{1 - \sin^2 x}$$

$$= \sqrt{\lim_{x\to 0} (1 - \sin^2 x)} = \sqrt{\lim_{x\to 0} 1 - \Big(\underbrace{\boxed{\lim_{x\to 0} \sin x}}_{=0}\Big)^2}$$

$$= \sqrt{1 - 0^2} = 1 \quad \text{— (ii)}$$

#

Cor = Corollary – 推論

(i) $\lim_{x \to c} \sin x = \sin c$

(ii) $\lim_{x \to c} \cos x = \cos c$

pf

let $x = c + h$: $x \to c \Leftrightarrow h \to 0$

(i)

$$\lim_{x \to c} \sin x \overset{\downarrow}{=} \lim_{h \to 0} \sin(c+h)$$

和角公式

some numbers

$$= \lim_{h \to 0} \left(\underline{\sin c} \cdot \cosh + \underline{\cos c} \cdot \sinh \right)$$

by Thm

$$= \sin c \cdot \underbrace{\left(\lim_{h \to 0} \cosh \right)}_{=1} + \cos c \cdot \underbrace{\left(\lim_{h \to 0} \sinh \right)}_{=0}$$

$$= (\sin c) \cdot 1 + (\cos c) \cdot 0 = \sin c$$

(ii)

$$\lim_{x \to c} \cos x = \lim_{h \to 0} \cos(c+h)$$

$$= \lim_{h \to 0} \left(\cos c \cdot \cosh - \sin c \cdot \sinh \right)$$

$$= \cos c \cdot \underbrace{\boxed{\lim_{h\to 0} \cosh}}_{=1} - \sin c \cdot \underbrace{\boxed{\lim_{h\to 0} \sinh}}_{=0}$$

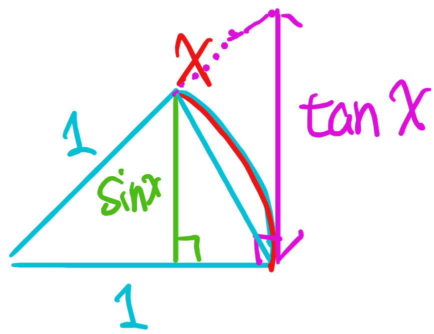
$$= \cos c \qquad \#$$

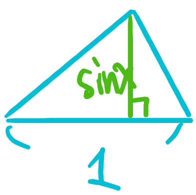
Thm (Eq. (2.5.6))

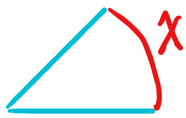
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

pf

① Assume $x > 0$, $x < \frac{\pi}{2}$.

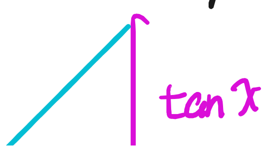


area(

) $\le$ area(

)

$\frac{1}{2} r^2 \theta = \frac{1}{2} x$

$\le$ area(

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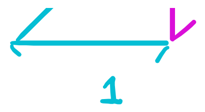
$\le$ area()

$$\Rightarrow \quad \frac{1}{2}\cdot \sin x \le \frac{1}{2}x \le \frac{1}{2}\tan x = \frac{1}{2}\frac{\sin x}{\cos x}$$

$$\Rightarrow \quad \frac{\sin x}{x} \le 1 \qquad\qquad \Rightarrow \quad \cos x \le \frac{\sin x}{x}$$

So

$$\cos x \le \frac{\sin x}{x} \le 1$$

Since $\lim_{x\to 0^+} \cos x = \cos 0 = 1 = \lim_{x\to 0^+} 1$,

by Pinching Thm,

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

One has a similar inequality when $x<0$. and can prove

$$\lim \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

So $$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \#$$

<u>Example</u>

$$\lim_{x \to 0} \frac{\sin(3x)}{2x} = \frac{1}{2} \cdot \lim_{x \to 0} \frac{\sin(3x)}{x}$$

Let $y = 3x \Rightarrow x = \frac{y}{3}$

$$= \frac{1}{2} \lim_{y \to 0} \frac{\sin y}{\frac{y}{3}} = \frac{3}{2} \boxed{\lim_{y \to 0} \frac{\sin y}{y}}$$

$1 =$

$$= \frac{3}{2} \qquad \#$$