

Calculus 9/14

Thm

= Theorem = 定理 = 對的事

if and only if 若且唯若

$$\textcircled{1} \quad \lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \rightarrow c^-} f(x) = L$$

and

$$\lim_{x \rightarrow c^+} f(x) = L$$

$\textcircled{2} \quad \lim_{x \rightarrow c} f(x)$ may or may not exist.

If $\lim_{x \rightarrow c} f(x)$ exists, then $\lim_{x \rightarrow c} f(x)$ is unique.

Limit laws

Thm (Thm 2.3.2)

If $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist,

and if k and c are fixed numbers,

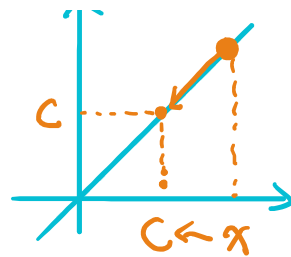
then

$f(x) = x$



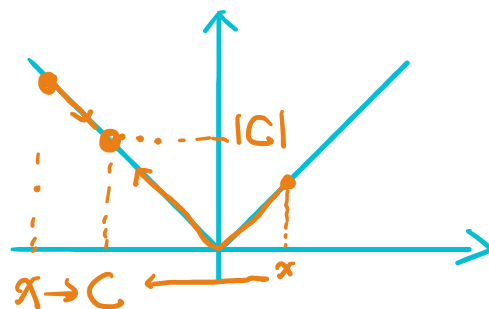
limit

$$\textcircled{1} \lim_{x \rightarrow c} x = c$$

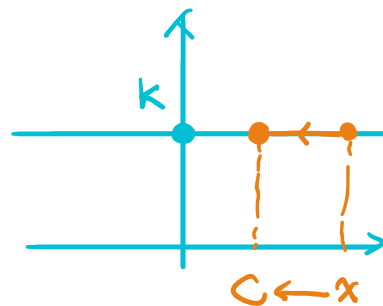


$$f(x) = |x|$$

$$\textcircled{2} \lim_{x \rightarrow c} |x| = |c|$$



$$\textcircled{3} \lim_{x \rightarrow c} k = k$$



$$\textcircled{4} \lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

e.g. $\lim_{x \rightarrow 1} \underline{\underline{2x}} \stackrel{\textcircled{4}}{=} \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} x$

$$\stackrel{\textcircled{1}}{=} 1 + 1 = 2 = 2 \cdot \left(\lim_{x \rightarrow 1} x \right)$$

(E) \cap (1, ∞) \cap \cap

$$\cup \lim_{x \rightarrow C} (k \cdot f(x)) = k \cdot \lim_{x \rightarrow C} f(x)$$

$$\textcircled{6} \lim_{x \rightarrow C} (f(x) \cdot g(x)) = \left(\lim_{x \rightarrow C} f(x) \right) \cdot \left(\lim_{x \rightarrow C} g(x) \right)$$

Example

$$\textcircled{a} \lim_{x \rightarrow 3} x^2 = \lim_{x \rightarrow 3} x \cdot x$$

$$\stackrel{\textcircled{6}}{=} \left(\lim_{x \rightarrow 3} x \right) \cdot \left(\lim_{x \rightarrow 3} x \right)$$

$$\stackrel{\textcircled{1}}{=} 3 \cdot 3 = 9 \quad \#$$

$$\textcircled{b} \lim_{x \rightarrow 3} x^2 - x = \lim_{x \rightarrow 3} x^2 + (-1) \cdot x$$

$$\stackrel{\textcircled{4}}{=} \boxed{\lim_{x \rightarrow 3} x^2} + \lim_{x \rightarrow 3} (-1) \cdot x$$

$\stackrel{\textcircled{a}}{=} 9$

$$= (-1) \cdot \left(\lim_{x \rightarrow 3} x \right) \stackrel{\textcircled{1}}{=} (-1) \cdot 3 = -3$$

$$= 9 - 3 = 6 \quad \#$$

$$\textcircled{c} \quad \lim_{x \rightarrow -2} |x| \cdot (x^3 + 5x)$$

$$\stackrel{\textcircled{6}}{=} \underbrace{\left(\lim_{x \rightarrow -2} |x| \right)}_{\textcircled{2}} \cdot \underbrace{\left(\lim_{x \rightarrow -2} x \cdot (x^2 + 5) \right)}_{\textcircled{6}}$$

$$|-2| = 2 \quad \left(\lim_{x \rightarrow -2} x \right) \cdot \left(\lim_{x \rightarrow -2} x^2 + 5 \right)$$

$$\stackrel{\textcircled{1} + \textcircled{4}}{=} (-2) \cdot \left(\lim_{x \rightarrow -2} \overbrace{x^2}^{x \cdot x} + \lim_{x \rightarrow -2} 5 \right)$$

$$\stackrel{\textcircled{2}}{=} (-2) \left((-2) \cdot (-2) + 5 \right) = -12$$

③

$$\text{So } \lim_{x \rightarrow -2} |x|(x^3 + 5x) = 2 \cdot (-18) = -36 \quad \#$$

Remark (§ 2.3)

Let $k_1, \dots, k_n, a_0, \dots, a_n$ be fixed numbers. Assume $\lim_{x \rightarrow c} f(x)$, $\lim_{x \rightarrow c} f_n(x)$ and $\lim_{x \rightarrow c} g(x)$ exist. Then

$$\textcircled{7} \quad \lim_{x \rightarrow c} (f(x) - g(x)) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)$$

$$\textcircled{8} \quad \lim_{x \rightarrow c} (k_1 \cdot f_1(x) + k_2 \cdot f_2(x) + \dots + k_n \cdot f_n(x)) \\ = k_1 \left(\lim_{x \rightarrow c} f_1(x) \right) + k_2 \left(\lim_{x \rightarrow c} f_2(x) \right) + \dots + k_n \left(\lim_{x \rightarrow c} f_n(x) \right)$$

e.g. $\lim_{x \rightarrow 1} (1 + |x| + x - x^2) = \lim_{x \rightarrow 1} 1 + \lim_{x \rightarrow 1} |x| + \lim_{x \rightarrow 1} x - \lim_{x \rightarrow 1} x^2$

$$= 2$$

⑨ $\lim_{x \rightarrow c} (f_1(x) \cdot f_2(x) \cdot \dots \cdot f_n(x))$

$$= \left(\lim_{x \rightarrow c} f_1(x) \right) \cdot \left(\lim_{x \rightarrow c} f_2(x) \right) \cdot \dots \cdot \left(\lim_{x \rightarrow c} f_n(x) \right)$$

e.g. $\lim_{x \rightarrow -1} x^8 = \left(\lim_{x \rightarrow -1} x \right)^8 = (-1)^8 = 1$

⑩ $\lim_{x \rightarrow c} (a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0)$ polynomial 多項式

$$= a_n c^n + a_{n-1} c^{n-1} + \dots + a_1 c + a_0$$

⑪ $\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$ if $C > 0$
in $n \rightarrow \infty$

屬於

“ε/11”

positive integers 正整數

Quotient (§2.3)

Assume $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist

Question:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = ?$$

There are 3 possible cases:

Case 1 (Thm 2.3.8)

If $\lim_{x \rightarrow c} g(x) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$$

eg. $\lim_{x \rightarrow 1} \frac{|x|}{x+1} = \frac{\lim_{x \rightarrow 1} |x|}{\lim_{x \rightarrow 1} x+1} = \frac{1}{2}$

$(x+1) \rightarrow 1+1=2 \neq 0$

Case 2 (Thm 2.3.10)

If $\lim_{x \rightarrow c} g(x) = 0$ and $\lim_{x \rightarrow c} f(x) \neq 0$,

then $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ does NOT exist

eg. $\lim_{x \rightarrow 1} \frac{|x|}{x-1}$ does NOT exist

$|x| \rightarrow 1=1 \neq 0$

$(x-1) \rightarrow 1-1=0$

Case 3

If $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} f(x) = 0$,

then anything can happen!!

"Simplify $\frac{f(x)}{g(x)}$ before taking limit"

Finish

Examples

(a) case 1 $\lim_{x \rightarrow 2} \frac{1}{x^3 - 1} = \frac{1}{7} \neq 0$

Annotations: The numerator '1' is circled in orange with an arrow pointing to '1'. The denominator 'x³-1' is circled in orange with an arrow pointing to '2³-1 = 7 ≠ 0'.

(b) case 2 $\lim_{x \rightarrow 3} \frac{x}{x^2 - 9}$

Annotations: The numerator 'x' is circled in orange with an arrow pointing to '3 ≠ 0'. The denominator 'x²-9' is circled in orange with an arrow pointing to '3²-9 = 0'.

does NOT exist.

(c) case 3 $\lim_{x \rightarrow 1} \frac{|x|(x-1)}{x-1}$

Annotations: The numerator is underlined in green. The denominator 'x-1' is circled in orange and crossed out with a green line, with an arrow pointing to '1-1 = 0'. Above the fraction, an arrow points to '1 · (1-1) = 0'.

$= \lim_{x \rightarrow 1} |x| = |1| = 1 \neq 0$

(d) case 3 $\lim_{x \rightarrow 1} \frac{x-1}{(x-1)^2}$

Annotations: The numerator 'x-1' is circled in green and crossed out with a green line, with an arrow pointing to '1-1 = 0'. The denominator '(x-1)²' is underlined in orange with an arrow pointing to '(1-1)² = 0'.

$= \lim_{x \rightarrow 1} \frac{1}{x-1}$

Annotations: The numerator '1' is circled in purple with an arrow pointing to '1 ≠ 0 case 2'. The denominator 'x-1' is underlined in purple with an arrow pointing to '1-1 = 0'.

does NOT exist. \neq

(e) case 3 $\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$

$x-9 \rightarrow 9-9=0$
 $\sqrt{x}-3 \rightarrow \sqrt{9}-3=0$

$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x})^2 - 3^2}{\sqrt{x}-3}$

$a^2 - b^2 = (a+b)(a-b)$

$\lim_{x \rightarrow 9} \frac{(\sqrt{x}+3)(\sqrt{x}-3)}{\sqrt{x}-3}$

$= \lim_{x \rightarrow 9} (\sqrt{x} + 3)$

$= \lim_{x \rightarrow 9} \sqrt{x} + \lim_{x \rightarrow 9} 3$

$\sqrt{9} = 3$

$= 3 + 3 = 6 \neq$

Remark

If neither $\lim_{x \rightarrow c} f(x)$ nor $\lim_{x \rightarrow c} g(x)$

exists then similar to case 3

exists, when, similar ~ ~ ~

anything can happen.

Simplify before taking limit!

eg.

$$\lim_{h \rightarrow 0} \frac{1 + \frac{1}{h}}{2 + \frac{1}{h}} = \frac{h+1}{h} \rightarrow 1 \neq 0 \quad \text{limit doesn't exist!!}$$

(case 2)

$$= \frac{2h+1}{h} \rightarrow 1 \quad \text{limit doesn't exist!!}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h} + 1}{2\cancel{h} + 1}$$

$$= \lim_{h \rightarrow 0} \frac{h+1}{2h+1} \rightarrow 0+1=1$$

$$\rightarrow 2 \cdot 0 + 1 = 1 \neq 0$$

$$= \frac{1}{1} = 1 \quad \neq$$