Calculus 9/14

Thm = Theorem = 定理 = 對的事

if and only if 若且唯若

① $\lim_{x \to c} f(x) = L \overset{\|}{\Longleftrightarrow} \lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

② $\lim_{x \to c} f(x)$ may or may not exist.

If $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c} f(x)$ is unique.

Limit laws

Thm (Thm 2.3.2)

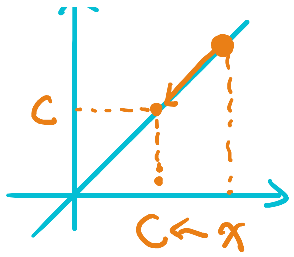
If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist,

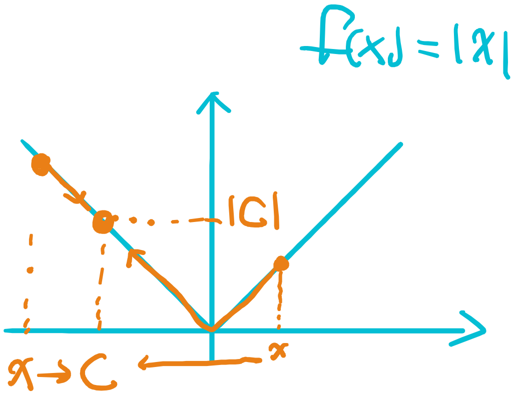
and if k and c are fixed numbers

$f(x)=x$

then

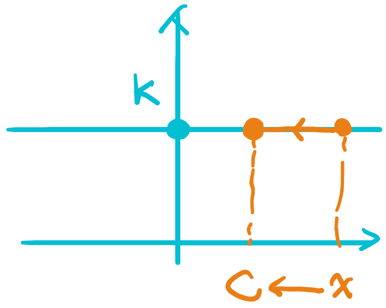
① $\lim_{x \to c} x = c$





② $\lim_{x \to c} |x| = |c|$

③ $\lim_{x \to c} k = k$



④ $\lim_{x \to c} \left(f(x) + g(x)\right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

e.g. $\lim_{x \to 1} (\underline{\overset{2x}{x+x}}) \overset{④}{=} \lim_{x \to 1} x + \lim_{x \to 1} x$

$\overset{①}{=} 1 + 1 = 2 = 2 \cdot \left(\lim_{x \to 1} x\right)$

⑤ $\lim \left(k\, f(x)\right)$

⑤ $\lim_{x\to C}(k\cdot f(x)) = k\cdot\lim_{x\to C} f(x)$

⑥ $\lim_{x\to C}(f(x)\cdot g(x)) = \left(\lim_{x\to C} f(x)\right)\cdot\left(\lim_{x\to C} g(x)\right)$

Example

ⓐ $\lim_{x\to 3} x^2 = \lim_{x\to 3} x\cdot x$

$\overset{\text{⑥}}{=} \left(\lim_{x\to 3} x\right)\cdot\left(\lim_{x\to 3} x\right)$

$\overset{\text{①}}{=} 3\cdot 3 = 9$ #

ⓑ $\lim_{x\to 3} x^2 - x = \lim_{x\to 3} x^2 + (-1)\cdot x$

$\overset{\text{④}}{=} \overset{\overset{\text{ⓐ}}{=}\, 9}{\boxed{\lim x^2}} + \lim (-1)\cdot x$

$x\to3$ $\qquad$ $x\to3$

$$= (-1)\cdot\left(\lim_{x\to3} x\right) \overset{①}{=} (-1)\cdot 3 = -3$$

$$= 9 - 3 = 6 \quad \#$$

ⓒ $\displaystyle\lim_{x\to -2} |x|\cdot(x^3+5x)$

$$\overset{⑥}{=} \underbrace{\left(\lim_{x\to-2}|x|\right)}_{\| ②} \cdot \underbrace{\left(\lim_{x\to-2} x\cdot(x^2+5)\right)}_{\| ⑥}$$

$$|-2| = 2 \qquad \left(\lim_{x\to-2} x\right)\cdot\left(\lim_{x\to-2} x^2+5\right)$$

$$\underset{①+④}{=} (-2)\cdot\left(\lim_{x\to-2} \overset{x\cdot x}{x^2} + \lim_{x\to-2} 5\right)$$

$$\overset{ⓐ}{=} (-2)\left((-2)\cdot(-2) + 5\right) = -18$$

③

So $\lim_{x\to -2} |x|(x^3+5x) = 2\cdot(-18) = -36$

#

Remark (§2.3)

Let $k_1, \dots, k_n, a_0, \dots, a_n$ be fixed numbers. Assume $\lim_{x\to c} f(x)$, $\lim_{x\to c} f_n(x)$ and $\lim_{x\to c} g(x)$ exist. Then

⑦ $$\lim_{x\to c}\left(f(x)-g(x)\right) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

⑧ $$\lim_{x\to c}\left(k_1\cdot f_1(x) + k_2\cdot f_2(x) + \cdots + k_n\cdot f_n(x)\right)$$

$$= k_1\cdot\left(\lim_{x\to c} f_1(x)\right) + k_2\cdot\left(\lim_{x\to c} f_2(x)\right) + \cdots + k_n\cdot\left(\lim_{x\to c} f_n(x)\right)$$

e.g. $$\lim_{x\to 1}(1+|x|+x-x^2)=\lim_{x\to 1}1+\lim_{x\to 1}|x|+\lim_{x\to 1}x-\lim_{x\to 1}x^2$$

$$=2$$

⑨ $$\lim_{x\to c}\left(f_1(x)\cdot f_2(x)\cdot\ldots f_n(x)\right)$$

$$=\left(\lim_{x\to c}f_1(x)\right)\cdot\left(\lim_{x\to c}f_2(x)\right)\cdots\left(\lim_{x\to c}f_n(x)\right)$$

e.g. $$\lim_{x\to -1}x^8=\left(\lim_{x\to -1}x\right)^8=(-1)^8=1$$

⑩ ↗ polynomial 多項式

$$\lim_{x\to c}\left(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right)$$

$$=a_nc^n+a_{n-1}c^{n-1}+\cdots+a_1c+a_0$$

⑪ $$\lim_{x\to c}\sqrt[n]{x}=\sqrt[n]{c}\quad\text{if}\quad c>0$$

in → $n \in \mathbb{N}$

屬於 $n \in \mathbb{N}$

positive integers 正整數

Quotient (§2.3)

Assume $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist

Question:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = ?$$

There are 3 possible cases:

Case 1 (Thm 2.3.8)

If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

e.g. $$\lim_{x\to 1} \frac{|x|}{\boxed{x+1}} = \frac{\lim_{x\to 1} |x|}{\lim_{x\to 1} x+1} = \frac{1}{2}$$

$\boxed{x+1} \rightarrow 1+1=2 \neq 0$

<u>case2</u> (Thm 2.3.10)

If $\lim_{x\to c} g(x)=0$ and $\lim_{x\to c} f(x) \neq 0$,

then $\lim_{x\to c} \frac{f(x)}{g(x)}$ does NOT exist

e.g. $$\lim_{x\to 1} \frac{|x|}{\boxed{x-1}} \quad \text{does NOT exist}$$

$|x| \rightarrow |1| = 1 \neq 0$

$\boxed{x-1} \rightarrow 1-1=0$

<u>case3</u>

If $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = 0$,

then anything can happen!!

"Simplify $\frac{f(x)}{g(x)}$ before taking limit"

Examp

Example

(a) case 1

$$\lim_{x\to 2} \frac{1}{x^3-1} = \frac{1}{7} \quad \#$$

$1 \to 1$

$x^3-1 \to 2^3-1 = 7 \neq 0$

(b) case 2

$$\lim_{x\to 3} \frac{x}{x^2-9}$$

$x \to 3 \neq 0$

$x^2-9 \to 3^2-9 = 0$

does NOT exist.

(c) case 3

$$\lim_{x\to 1} \frac{|x|\cancel{(x-1)}}{\cancel{x-1}}$$

$\to |1|\cdot(1-1) = 0$

$\to 1-1 = 0$

$$= \lim_{x\to 1} |x| = |1| = 1 \quad \#$$

(d) case 3

$$\lim_{x\to 1} \frac{\cancel{x-1}}{(x-1)^{\cancel{2}}}$$

$1-1=0$

$\to (1-1)^2 = 0$

$$= \lim_{x\to 1} \frac{1}{x-1}$$

$\to 1 \neq 0$ case 2

$\to 1-1 = 0$

does NOT exist. #

(e)

case3 $\lim_{x\to 9} \dfrac{\boxed{x-9}}{\boxed{\sqrt{x}-3}}$

$x-9 \rightarrow 9-9=0$

$x-9 = (\sqrt{x})^2 - 3^2$

$\sqrt{x}-3 \rightarrow \sqrt{9}-3=0$

a^2-b^2
$= (a+b)(a-b)$

$$= \lim_{x\to 9} \frac{(\sqrt{x}+3)\cancel{(\sqrt{x}-3)}}{\cancel{\sqrt{x}-3}}$$

$$= \lim_{x\to 9}(\sqrt{x}+3)$$

③ $= 3$

$$= \lim_{x\to 9}\sqrt{x} + \lim_{x\to 9} 3$$

⑪ $=$ $\sqrt{9} = 3$

$$= 3+3 = 6 \quad \#$$

Remark

If neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$

exists then similar to case 3

exists, then, similar to case,

anything can happen.

Simplify before taking limit!

eg.

$$\lim_{h\to 0}\frac{1+\frac{1}{h}}{2+\frac{1}{h}}$$

$= \dfrac{h+1}{h}$ $\quad h+1 \to 1 \neq 0$, $\quad h \to 0$

limit doesn't exist!!

(case 2)

$= \dfrac{2h+1}{h}$ $\quad 2h+1 \to 1$, $\quad h \to 0$

limit doesn't exist!!

$$= \lim_{h\to 0}\frac{\frac{h+1}{\cancel{h}}}{\frac{2h+1}{\cancel{h}}}$$

$$= \lim_{h\to 0}\frac{h+1}{2h+1}$$

$h+1 \to 0+1=1$

$2h+1 \to 2\cdot 0+1=1 \neq 0$

$$= \frac{1}{1} = 1 \quad \#$$