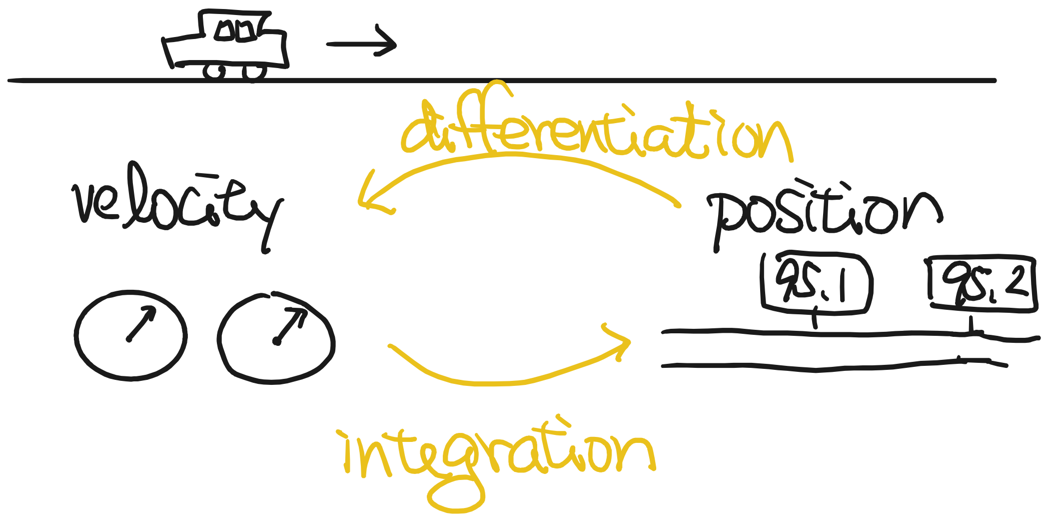
Calculus 9/12

Introduction

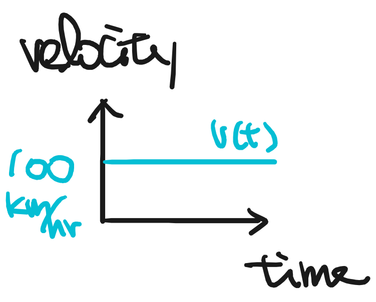
Main topics : "differentiation" and "integration"

Consider

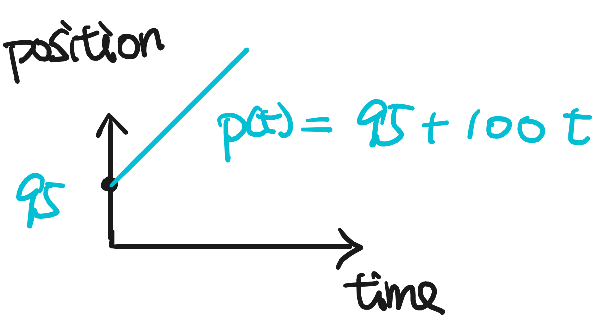


e.g.

(i)

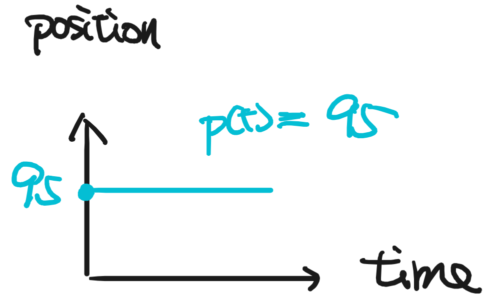


$\Rightarrow$

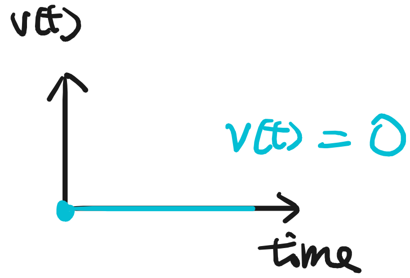


$$95 + \int_0^t 100 \, ds = 95 + 100t$$

(ii)



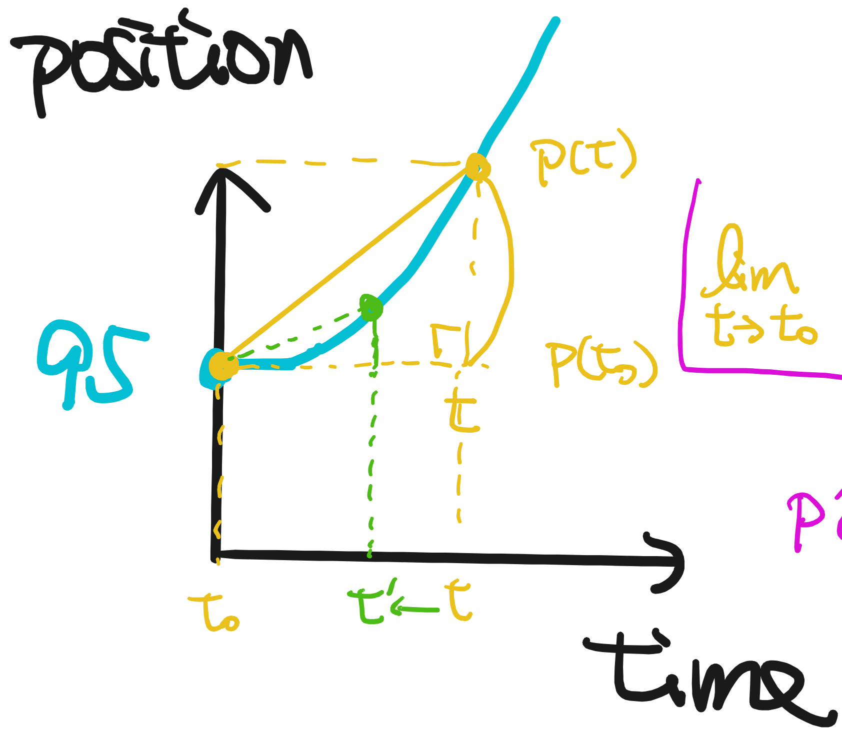
$\Rightarrow$



$$\frac{d}{dt}(95) = 0$$

$$p(t) = 95 + 100\, t e^{-t}$$

(iii)



approximation of $v(t_0)$

$$\boxed{\lim_{t \to t_0} \text{“}\frac{p(t) - p(t_0)}{t - t_0}\text{”}}$$

$$= p'(t_0) = \left.\frac{d}{dt}\right|_{t=t_0} p(t) = v(t_0)$$

Main problem: limit

§ Limit

Intuitive examples:

We will see 3 types of limits in Calculus:

- Function type A:

$$\lim_{x \to c} f(x) \qquad (c: \text{fixed number})$$

- Function type B:

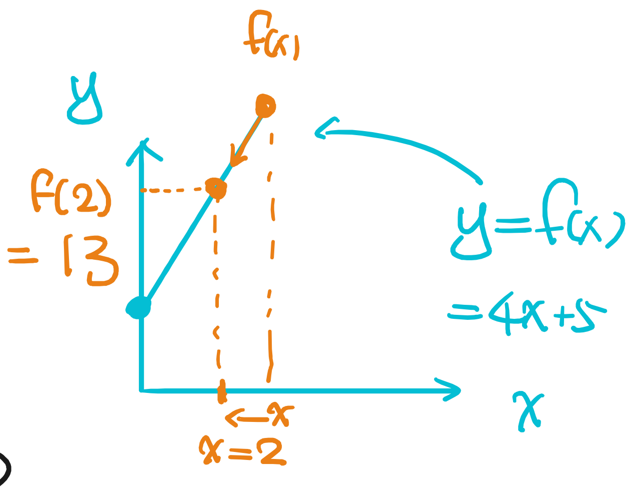
$$\lim_{x \to \infty} f(x) \quad \text{or} \quad \lim_{x \to -\infty} f(x)$$

$(\infty = \text{infinity})$

- Sequence type: $\lim_{n \to \infty} a_n$

Example

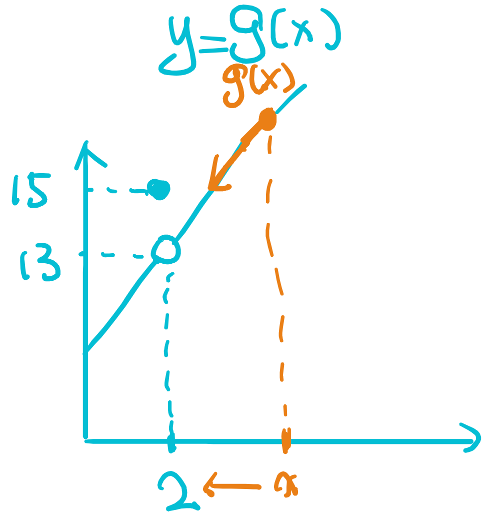
① $f(x) = 4x + 5$



$\lim f(x) = f(2) = 13$

$$\lim_{x\to 2} f(x) = f(2) = 13$$

② $g(x) = \begin{cases} 4x+5, & x \neq 2 \\ 15, & x = 2 \end{cases}$



$$\lim_{x\to 2} g(x) = 13$$

$$\neq 15 = g(2)$$

③ $$f(x) = \frac{x^2-9}{x-3}$$

Note that $f(x)$ is NOT defined at

$x = 3$: $f(3) = \dfrac{3^2-9}{3-3} = \dfrac{0}{0} = ??$

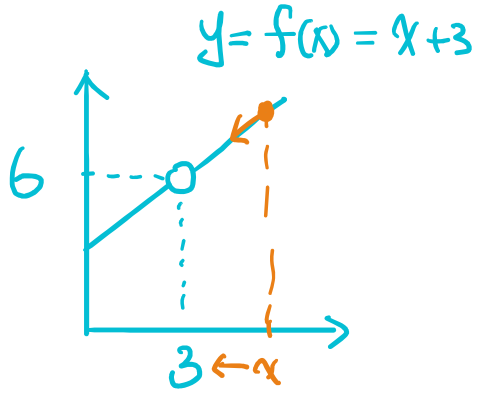
$a^2-b^2 = (a+b)(a-b)$

However, if $x \neq 3$

$$f(x) = \frac{x^2-9}{x-3} = \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}$$

$9 = 3^2$

$$= x + 3$$

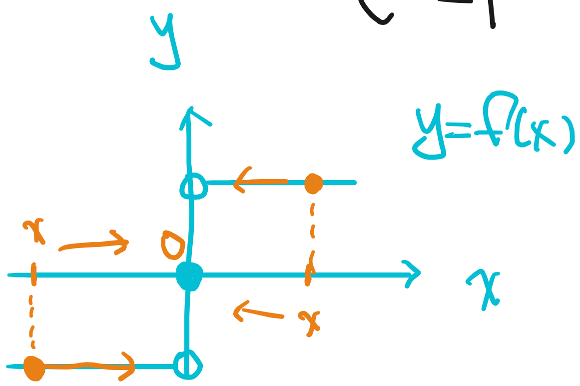


$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3)$$

$$= 6$$

④

$$f(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$



$\lim_{x\to 0} f(x)$ does NOT exist

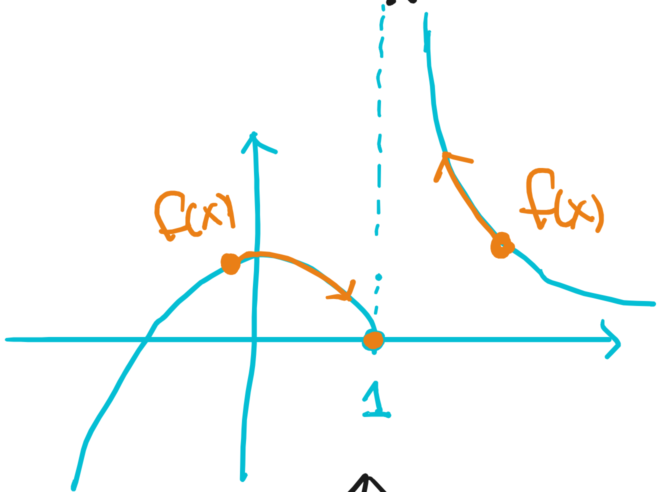
We say $\lim_{x\to 0^+} f(x) = +1$ (右極限) ※

$\lim_{x\to 0^-} f(x) = -1$ (左極限)

(5)

(5)

$$f(x) = \begin{cases} 1-x^2 & x \le 1 \\ \dfrac{1}{x-1} & x > 1 \end{cases}$$



$\lim_{x \to 1^-} f(x) = 0$

$\lim_{x \to 1^+} f(x)$ does NOT exist

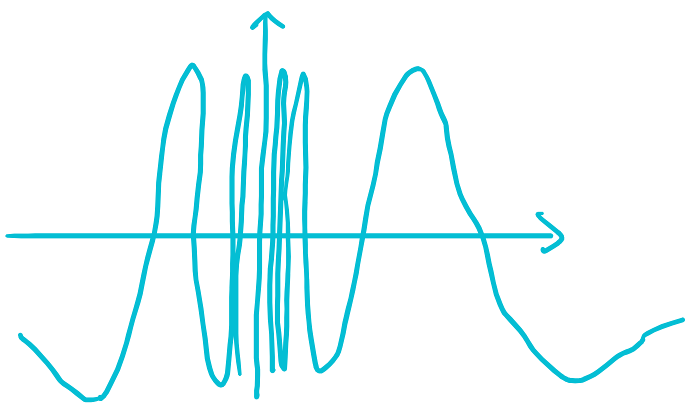
or

$\lim_{x \to 1^+} f(x) = +\infty$

$\lim_{x \to 1} f(x)$ does NOT exist

(6)

$$f(x) = \sin\left(\frac{\pi}{x}\right)$$



$\lim_{x \to 0} f(x)$, $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$ do NOT exist

§ Precise definitions of limits

(§2.2)

Definition = 定義

<u>Def</u> (Def 2.2.1, 2.2.7, 2.2.8)

Let f be a function defined on

開區間

on $(c-p, c) \cup (c, c+p)$, $p>0$,

union

we say

for all   there exists   such that

① $\boxed{\lim_{x\to c} f(x) = L}$ if $\forall \varepsilon>0$ $\exists \delta>0$ s.t.

if $0<|x-c|<\delta$, then $|f(x)-L|<\varepsilon$

② $\boxed{\lim_{x\to c^-} f(x) = L}$ if $\forall \varepsilon>0$ $\exists \delta>0$ s.t.

if $c-\delta<x<c$, then $|f(x)-L|<\varepsilon$

③ $\boxed{\lim_{x\to c^+} f(x) = L}$ if $\forall \varepsilon>0$ $\exists \delta>0$ s.t.

if $c<x<c+\delta$, then $|f(x)-L|<\varepsilon$

Example

Example

Prove $\lim_{x \to 2} f(x) = 13$, where

$$f(x) = \begin{cases} 4x+5, & x \neq 2 \\ 15, & x = 2 \end{cases}$$

pf

Given any $\varepsilon > 0$, we choose $\delta = \frac{\varepsilon}{5} > 0$.

If $\underbrace{0 < |x-2|}_{\Rightarrow x \neq 2} < \frac{\varepsilon}{5}$, then

$$|f(x) - 13| = |4x+5-13|$$

$$= |4x-8| = 4|x-2| < 4 \cdot \frac{\varepsilon}{5}$$

$$< \varepsilon \qquad \#$$