Calculus — Homework 8 (Fall 2023)

1. Evaluate the integral.

   (a) $\displaystyle\int_0^1 (2x-3)\,dx.$

   (b) $\displaystyle\int_1^2 (2x+x^2)\,dx.$

   (c) $\displaystyle\int_1^4 2\sqrt{x}\,dx.$

   (d) $\displaystyle\int_0^1 (x^{3/4}-2x^{1/2})\,dx.$

   (e) $\displaystyle\int_0^1 (x+1)^{17}\,dx.$

   (f) $\displaystyle\int_1^2 \frac{6-x}{x^3}\,dx.$

   (g) $\displaystyle\int_1^2 2x(x^2+1)\,dx.$

   (h) $\displaystyle\int_0^{\pi/2} \cos x\,dx.$

   (i) $\displaystyle\int_0^{2\pi} \sin x\,dx.$

   (j) $\displaystyle\int_{-\pi/6}^{\pi/6} \sin x\cos x\,dx.$

   (k) $\displaystyle\int_2^5 (x-3)\,dx.$

   (l) $\displaystyle\int_2^5 |x-3|\,dx.$

2. Let f be a function such that f' is continuous on $[a,b]$. Show that

$$\int_a^b f(x)f'(x)\,dx = \frac{1}{2}\Big[\big(f(b)\big)^2 - \big(f(a)\big)^2\Big].$$

3. Calculate.

   (a) $\displaystyle\int_0^1 x(x^2+1)^3\,dx.$

   (b) $\displaystyle\int_{-1}^0 3x^2(4+2x^3)^2\,dx.$

   (c) $\displaystyle\int_0^1 x\sqrt{x+1}\,dx.$

   (d) $\displaystyle\int_0^1 \frac{x+3}{\sqrt{x+1}}\,dx.$

   (e) $\displaystyle\int_{-\pi}^{\pi} \sin^4 x\cos x\,dx.$

   (f) $\displaystyle\int_0^1 \cos^2(\frac{\pi}{2}x)\sin(\frac{\pi}{2}x)\,dx.$

   (g) $\displaystyle\int_0^{\pi} x\cos x^2\,dx.$

   (h) $\displaystyle\int_0^{\pi} x^2\cos x\,dx.$

   (i) $\displaystyle\int_0^{\pi/2} \cos^2 2x\,dx.$

   (j) $\displaystyle\int_0^{2\pi} \sin^2 x\,dx.$

   (k) $\displaystyle\int_0^1 x(x+5)^{14}\,dx.$

   (l) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx.$

   (m) $\displaystyle\int_0^{\pi/2} \cos(\sqrt{x})\,dx.$

4. Calculate.

   (a) $\displaystyle\frac{d}{dx}\left(\int_0^{1+x^2} \frac{dt}{\sqrt{2t+5}}\right).$

   (b) $\displaystyle\frac{d}{dx}\left(\int_{3x}^{1/x} \cos 2t\,dt\right).$

   (c) $\displaystyle\int_{-3}^3 \frac{t^3}{1+t^2}\,dt.$

   (d) $\displaystyle\int_{-\pi/4}^{\pi/4} (x^2-2x+\sin x+\cos 2x)\,dx.$

5. Let f be a continuous function, and a,b,c be real numbers.

   (a) Show that

$$\int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(x)\,dx.$$

   (b) Show that, if $c\neq 0$,

$$\frac{1}{c}\int_{ac}^{bc} f(x/c)\,dx = \int_a^b f(x)\,dx.$$

6. Let f be continuous on $[-a, a]$.

   (a) Show that
$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx.$$

   (b) Show that
$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) + f(-x)]\,dx.$$

7. True or false. Explain your answers. Assume f and g are continuous on $[a, b]$, $a < b$.

   (a) If $\displaystyle\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $\displaystyle\int_a^b [f(x) - g(x)]\,dx > 0$.

   (b) If $\displaystyle\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $f(x) > g(x)$ for all $x \in [a, b]$.

   (c) If $\displaystyle\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $f(x) > g(x)$ for some $x \in [a, b]$.

   (d) If $\displaystyle\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then $\displaystyle\int_a^b |f(x)|\,dx > \int_a^b |g(x)|\,dx$.

   (e) If $\displaystyle\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

   (f) If $\displaystyle\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for some $x \in [a, b]$.

   (g) If $\displaystyle\int_a^b |f(x)|\,dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

   (h) If $\displaystyle\int_a^b f(x)\,dx = 0$, then $\displaystyle\int_a^b |f(x)|\,dx = 0$.

8. Suppose that f is continuous on $[a, b]$, $a < b$, and $\displaystyle\int_a^b f(x)\,dx = 0$. Prove that there is at least one number c in (a, b) for which $f(c) = 0$.