Calculus — Homework 6 (Fall 2023)

1. Express dy/dx in terms of x and y.

   (a) $x^2 + y^2 = 4$.

   (b) $x^3 + y^3 - 3xy = 0$.

   (c) $\sin(x + y) = xy$.

   (d) $\sqrt{x} + \sqrt{y} = 4$, $x, y > 0$.

2. Find equations for the tangent line at the point indicated.

   (a) $9x^2 + 4y^2 = 72$; $(2, 3)$.

   (b) $x^2 + xy + 2y^2 = 28$; $(-2, -3)$.

   (c) $x = \cos y$; $(\frac{1}{2}, \frac{\pi}{3})$.

3. (i) Determine whether or not f satisfies the conditions of the mean value theorem on the indicated interval $[a, b]$. (ii) Find all the numbers c such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

   (a) $f(x) = x^3 - x$; $[0, 1]$.

   (b) $f(x) = x^2$; $[1, 2]$.

   (c) $f(x) = 3\sqrt{x} - 4x$; $[1, 4]$.

   (d) $f(x) = \sin x$; $[0, \pi]$.

   (e) $f(x) = \sqrt{1 - x^2}$; $[0, 1]$.

   (f) $f(x) = x^{2/3} - 1$; $[-1, 1]$.

4. Suppose that f is differentiable on $(2, 6)$ and continuous on $[2, 6]$. Given that $1 \le f'(x) \le 3$ for all x in $(2, 6)$, show that
$$4 \le f(6) - f(2) \le 12.$$

5. Prove that for all real numbers x and y

   (a) $|\cos x - \cos y| \le |x - y|$.

   (b) $|\sin x - \sin y| \le |x - y|$.

6. Suppose that f'' is continuous on $[a, b]$ and that f has 3 distinct zeros in $[a, b]$. Prove that f'' has at least one zero in (a, b).

7. Find the intervals on which f is increasing and the intervals on which f is decreasing.

   (a) $f(x) = x^3 - 3x + 2$.

   (b) $f(x) = x + \frac{1}{x}$.

   (c) $f(x) = |x^2 - 5|$.

   (d) $f(x) = x - \cos x$.

8. Show that
$$\tan x > x,$$
for all x in $(0, \frac{\pi}{2})$.

9. True or false? Explain your answers.

   (a) The function $f(x) = x^2$ is an increasing function on $(-\infty, \infty)$.

   (b) The function $f(x) = x^2$ is a decreasing function on $(-\infty, \infty)$.

   (c) The function $f(x) = x^2$ is an increasing function on $(0, \infty)$.

   (d) The function $f(x) = x^3$ is an increasing function on $(-\infty, \infty)$.

10. Suppose a function f has derivative
$$f'(x) = x^3(x - 1)^2(x + 1)(x - 2).$$
At what numbers x, if any, does f have a local maximum? A local minimum?

11. Find the critical points, local maximums and local minimums of f.

(a) $f(x) = x^3 - 3x + 2$.

(b) $f(x) = x + \frac{1}{x}$.

(c) $f(x) = |x^2 - 5|$.

(d) $f(x) = x - \cos x$.