Calculus — Homework 6.5 (Fall 2023)

1. True or false? Explain your answers.
   (a) The equation $6x^4 - 7x + 1 = 0$ has 4 distinct real roots.
   (b) The equation $x^5 + 13x + 1 = 0$ has exactly one real root.

Solution:

(a) F (b) T

2. Find the critical points. Then find and classify all the extreme values.
   (a) $f(x) = x^2 - 4x + 1, \quad 0 \le x \le 3.$
   (b) $f(x) = \dfrac{x^2}{1+x^2}, \quad -1 \le x \le 2.$
   (c) $f(x) = \sin 2x - x, \quad 0 \le x \le \pi.$
   (d) $f(x) = 1 - \sqrt[3]{x-1}, \quad x \in (-\infty, \infty).$
   (e) $f(x) = \begin{cases} x^2 + 2x + 2, & x < 0, \\ x^2 - 2x + 2, & 0 \le x \le 2. \end{cases}$

Solution:

(a) critical point: $x = 2$. endpoint: $x = 0, 3$.
$f(0) = 1$ is a global maximum, $f(2) = -3$ is a global minimum,
$f(3) = -2$ is a local maximum.

(b) critical point: $x = 0$, endpoint: $x = -1, 2$.
$f(-1) = \frac{1}{2}$ is a local maximum, $f(0) = 0$ is a global minimum,
$f(2) = \frac{4}{5}$ is a global maximum.

(c) critical point: $x = \frac{\pi}{6}, \frac{5\pi}{6}$, endpoint: $x = 0, \pi$.
$f(0) = 0$ is a local minimum, $f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ is a global maximum,
$f(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}$ is global minimum, $f(\pi) = -\pi$ is a local maximum.

(d) critical point: $x = 1$. No maximum, no minimum.

(e) critical point: $x = -1, 0, 1$, endpoint: $x = 2$.
global minimum: $f(-1) = f(1) = 1$, local maximum: $f(0) = f(2) = 2$.

3. Find the greatest possible value of xy given that x and y are both positive and $x + y = 40$.

Solution: 400

4. Describe the concavity of the graph and find the points of inflection (if any).
   (a) $f(x) = x + \dfrac{1}{x}.$
   (b) $f(x) = x^3(1 - x).$
   (c) $f(x) = \sin^2 x, \quad 0 \le x \le \pi.$

Solution:

(a) concave down on $(-\infty, 0)$, concave up on $(0, \infty)$, no point of inflection

(b) concave down on $(-\infty, 0) \cup (\frac{1}{2}, \infty)$, concave up on $(0, \frac{1}{2})$
point of inflection: $(0, f(0)) = (0, 0)$, $(\frac{1}{2}, f(\frac{1}{2})) = (\frac{1}{2}, \frac{1}{16})$

(c) concave down on $(\frac{\pi}{4}, \frac{3\pi}{4})$, concave up on $(0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$
point of inflection: $(\frac{\pi}{4}, f(\frac{\pi}{4})) = (\frac{\pi}{4}, \frac{1}{2})$, $(\frac{3\pi}{4}, f(\frac{3\pi}{4})) = (\frac{3\pi}{4}, \frac{1}{2})$

5. Determine A and B so that the curve

$$y = A\cos 2x + B\sin 3x$$

has a point of inflection at $(1, 4)$.

Solution: $\text{A} = \dfrac{36}{5\cos 2}$, $\text{B} = -\dfrac{16}{5\sin 3}$.

6. Prove that if $|f(x) - (3x+2)| \leq |x|^{\frac{3}{2}}$ for any real numbers x, then f is differentiable at 0, and $y = 3x+2$ is the tangent line of the graph of f at $\big(0, f(0)\big)$. (Approximation of f by a line around $x = 0$.)

7. Calculate the following limits.

(a) $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$.

(b) $\displaystyle\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$.

(c) $\displaystyle\lim_{x\to 0} \frac{x+\sin(\pi x)}{x-\sin(\pi x)}$.

(d) $\displaystyle\lim_{x\to 0} \frac{\cos x - \cos 3x}{\sin(x^2)}$.

(e) $\displaystyle\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x}$.

(f) $\displaystyle\lim_{x\to 1} \frac{x^{1/2}-x^{1/4}}{x-1}$.

(g) $\displaystyle\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{\sin^2 x}$.

(h) $\displaystyle\lim_{x\to(\pi/2)^-} (x - \frac{\pi}{2})\sec x$.

Solution:

(a) 0

(b) $\frac{1}{4}$

(c) $\frac{1+\pi}{1-\pi}$

(d) 4

(e) $\frac{1}{\sqrt{2}}$

(f) $\frac{1}{4}$

(g) $-\frac{1}{3}$

(h) -1

8. Read the descriptions about upper (Riemann) sum and lower (Riemann) sum on the following websites:

- upper Riemann sum
- lower Riemann sum