Calculus — Homework 4.5 (Fall 2023)

1. Differentiate the following functions.

   (a) $f(x) = 1 - x$.

   (b) $f(x) = 11x^5 - 6x^3 + 8$.

   (c) $f(x) = \dfrac{3}{x^2}$.

   (d) $f(x) = (x^2 - 1)(x - 3)$.

   (e) $f(x) = \dfrac{x^3}{1 - x}$.

   (f) $f(x) = \Big(1 + \dfrac{1}{x}\Big)\Big(1 + \dfrac{1}{x^2}\Big)$.

Solution:

(a) -1

(b) $55x^4 - 18x^2$

(c) $-\dfrac{6}{x^3}$

(d) $3x^2 - 6x - 1$

(e) $\dfrac{-2x^3 + 3x^2}{(1 - x)^2}$

(f) $\Big(-\dfrac{1}{x^2}\Big)\Big(1 + \dfrac{1}{x^2}\Big) + \Big(1 + \dfrac{1}{x}\Big)\Big(-\dfrac{2}{x^3}\Big)$

2. Find the point(s) where the tangent line is horizontal.

   (a) $f(x) = (x - 2)(x^2 - x - 11)$.

   (b) $f(x) = x^2 - \dfrac{16}{x}$.

Solution:

(a) $(-1, 27)$, $(3, -5)$

(b) $(-2, 12)$

3. Find dy/dx, d^2y/dx^2 and d^3y/dx^3.

   (a) $y = \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 + x + 1$.

   (b) $y = (1 + 5x)^2$.

   (c) $y = x^3 - \dfrac{1}{x^3}$.

   (d) $y = \dfrac{x^4 + 2}{x}$.

Solution:

(a) $y' = x^2 + x + 1$, $y'' = 2x + 1$, $y''' = 2$

(b) $y' = 50x + 10$, $y'' = 50$, $y''' = 0$

(c) $y' = 3x^2 + 3x^{-4}$, $y'' = 6x - 12x^{-5}$, $y''' = 6 + 60x^{-6}$

(d) $y' = 3x^2 - 2x^{-2}$, $y'' = 6x + 4x^{-3}$, $y''' = 6 - 12x^{-4}$

4. Let p be an arbitrary polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0.$$

   (a) Find $(d^{n-1}/dx^{n-1})\big(p(x)\big)$ and $(d^n/dx^n)\big(p(x)\big)$.

   (b) What is $(d^k/dx^k)\big(p(x)\big)$ for $k > n$?

Solution:

(a) $(d^{n-1}/dx^{n-1})\big(p(x)\big) = a_n \cdot n! \cdot x + a_{n-1} \cdot (n-1)!$. $\quad (d^n/dx^n)\big(p(x)\big) = a_n \cdot n!$.

(b) 0

5. Let f, g be arbitrary differentiable functions.
   (a) Find a formula for $(f \cdot g)''(x)$.
   (b) Assume $g(x) \neq 0$. Find a formula for $(f/g)''(x)$.

Solution:

(a) $(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$

(b) $\dfrac{(f' \cdot g - f \cdot g')' \cdot g^2 - (f' \cdot g - f \cdot g')(g^2)'}{g^4} = \dfrac{(f'' \cdot g - f \cdot g'' - 2f' \cdot g') \cdot g^2 + 2fg(g')^2}{g^4}$

6. True or false? Explain your answers.
   (a) The derivative of f at c is the limit $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.
   (b) If a function f is differentiable at c, then f is continuous at c.
   (c) If a function f is continuous at c, then f is differentiable at c.

Solution:

(a) T $\quad$ (b) T $\quad$ (c) F