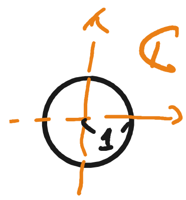
Linear Algebra 12/21

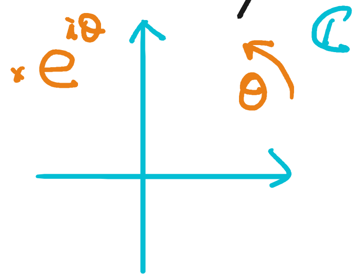
Group $\longleftrightarrow$ Symmetry

Lie group $\longleftrightarrow$ Smooth symmetry

e.g. a Lie group $\quad z = e^{i\theta}$

$U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \} \cong S^1 =$ 

acts on $\mathbb{C}$ by multiplication



$\leftrightsquigarrow$ polar coordinate

Today:

Closed subgroups of $GL_n(\mathbb{C})$

A big success of Lie theory:

$\exists$ nice correspondence between

$\left\{ \begin{matrix} \text{Lie} \\ \text{groups} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Lie} \\ \text{algebras} \end{matrix} \right\}$

vector + bilinear map s.t. ...

difficult much easier

Examples of matrix group

Regarding $M_n(\mathbb{C})$ as $\mathbb{R}^{2n^2}$, it is a metric space. Since

polynomial (variables $= a_{ij}$)

$$\det : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

$$\det(a_{ij}) = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

is a poly, in particular, continuous,

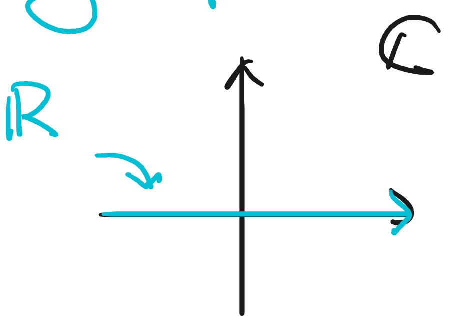
$$\underline{GL_n(\mathbb{C})} = \det^{-1}(\underbrace{\mathbb{C} \setminus \{0\}}_{\text{open in } \mathbb{C}})$$ ← metric space

is an open subset of $M_n(\mathbb{C})$

general linear group over $\mathbb{C}$

Example

① $\mathbb{R} \overset{\text{closed}}{\hookrightarrow} \mathbb{C}$



$M_n(\mathbb{R})$ is also closed in $M_n(\mathbb{C})$

subgp of $GL_n(\mathbb{C})$

$\Rightarrow GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$

$= GL_n(\mathbb{C}) \cap M_n(\mathbb{R})$ general linear group over $\mathbb{R}$

is closed in $GL_n(\mathbb{C})$ #

Recall: As a subspace $GL_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$, the closed subsets in $GL_n(\mathbb{C})$

$$= \{ GL_n(\mathbb{C}) \cap K \mid K \text{ is closed in } M_n(\mathbb{C}) \}$$

② The special linear group over $\mathbb{C}$ is

$$SL_n(\mathbb{C}) = \{ A \in GL_n(\mathbb{C}) \mid \det(A) = 1 \}$$

$$= GL_n(\mathbb{C}) \cap \underbrace{\det^{-1}(\{1\})}_{\subseteq M_n(\mathbb{C}) \text{ closed}}$$

which is a closed subgroup of $GL_n(\mathbb{C})$

For $A, B \in SL_n(\mathbb{C})$,

$$\det(AB) = \det(A)\det(B)$$
$$= 1 \times 1 = 1$$
$$\Rightarrow \quad A \cdot B \in SL_n(\mathbb{C})$$

Similarly,

$$SL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det(A) = 1 \}$$

$SL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det A = 1 \}$

$$= \det^{-1}(\{1\}) \cap M_n(\mathbb{R}) \cap GL_n(\mathbb{C})$$

is a closed subgroup of $GL_n(\mathbb{C})$

③ The <u>unitary group</u> of degree n #

is

$$U(n) = \{ A \in GL_n(\mathbb{C}) \mid \underbrace{AA^* = A^*A = I_n}_{\Leftrightarrow\ A^{-1} = A^*} \}$$

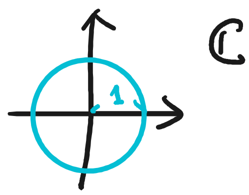
where A^* is the conjugate transpose of A

e.g.

$$A = \begin{pmatrix} i & 1-i \\ 1 & 2 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \bar{i} & \bar{1} \\ \overline{1-i} & \bar{2} \end{pmatrix}$$

$$= \begin{pmatrix} -i & 1 \\ 1+i & 2 \end{pmatrix}$$

e.g. $$U(1) = \{ \overset{z}{\overset{\shortparallel}{A}} \in GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\} \mid \overset{|z|^2}{\overset{\shortparallel}{\bar{z}z}} = z\bar{z} = 1 \}$$

$=$ 

A matrix in $U(n)$ is called a <u>unitary matrix</u>.

Note $U(1)$ is the ...

Note $U(n)$ is the solution set of

$$AA^* - I_n = 0$$

Assume $A = (a_{ij})$, $A^* = (\overline{a_{ji}})$

$$= \underbrace{\left(\sum_{k=1}^{n} a_{ik}\overline{a_{jk}}\right)_{i,j} - I_n}_{\text{Continuous}}$$

$U(n) = (\text{a continuous function})^{-1}(\{0\})$

is closed in $GL_n(\mathbb{C})$

$U(n)$ is a subgp because

$\forall A, B \in U(n)$,

$$(AB)^* = B^* \cdot A^*$$
$$= B^{-1} \cdot A^{-1} = (A \cdot B)^{-1}$$

Note
$(AB)^T = B^T A^T$
$\overline{AB} = \overline{A} \cdot \overline{B}$
$\Rightarrow (AB)^* = B^* A^*$

$\Rightarrow AB \in U(n)$ #

④ The orthogonal group in dim n is

$$O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = A^T A = I_n\}$$

$$= U(n) \cap GL_n(\mathbb{R})$$

which is also closed in $GL_n(\mathbb{C})$.

It is a subgp because
$\forall A, B \in O(n)$,
$$(AB)^T = B^T \cdot A^T = B^{-1} \cdot A^{-1} = (AB)^{-1}$$
$\Rightarrow AB \in O(n)$.

<u>Remark</u>

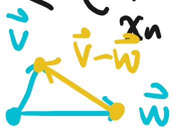
why? $\Updownarrow$ $A^TA = I$

$$O(n) = \left\{ \text{distance-preserving linear transformations} \right\}$$

Note that

(i) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, $\quad \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} x_i y_i$

(ii) $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$



(iii) distance-preserving $=$ $\langle\, , \rangle$-preserving in $\mathbb{R}^n$

(iv) $\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^T\vec{w} \rangle$

(v) A is $\langle\, , \rangle$-preserving

$\Leftrightarrow \langle A\vec{v}, A\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle \quad \forall\, \vec{v}, \vec{w} \in \mathbb{R}^n$
$\qquad \| \qquad\qquad\quad //$
$\quad \langle \vec{v}, A^TA\vec{w} \rangle$

$\Leftrightarrow A^TA = I_n$

$a = \cos\theta$
$b = -\sin\theta$
...

$$\begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$$
"1" $\quad$ $=1$

Case $n=2$:

$$O(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \underline{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

rotation

det = −1

$$= \left\{ \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\det = 1} , \underbrace{\begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}}_{\text{rotation} \times \text{reflection}} \,\middle|\, \theta \in \mathbb{R} \right\}$$

$$\theta = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

= reflection along $x = y$

NOT connected

The special orthogonal group (in dim n) is

$$SO(n) = O(n) \cap SL(n)$$

$$= \{ A \in O(n) \mid \det(A) \}$$

= the connected component of $O(n)$ which contains I_n

e.g.

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}$$

$$\cong \{ \cos\theta + i\sin\theta \mid \theta \in \mathbb{R} \} = U(1)$$

#

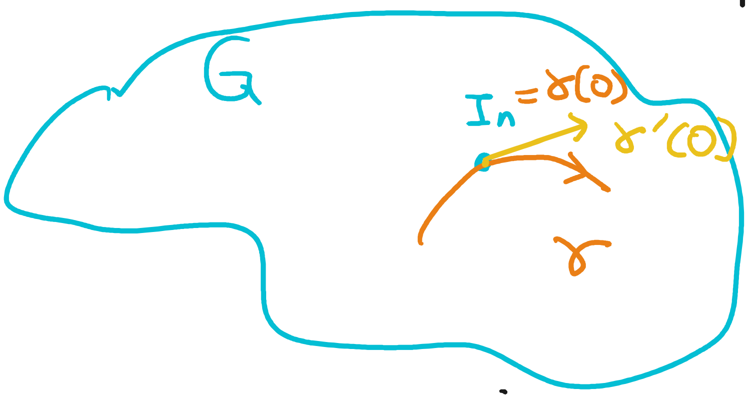
Lie algebra of a matrix group

Let G be a closed subgp of $GL_n(\mathbb{C})$

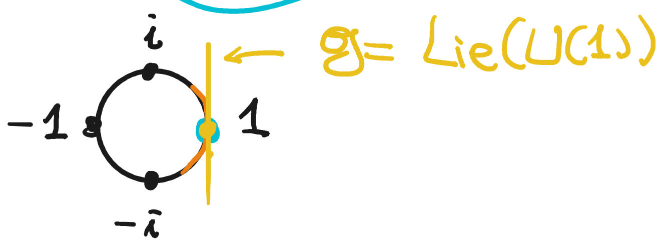
The Lie algebra associated with G is

$$\mathfrak{g} = \mathrm{Lie}(G) = T_{I_n} G$$

$$= \left\{ \gamma'(0) \in \underbrace{M_n(\mathbb{C})}_{\cong \mathbb{R}^{2n^2}} \;\middle|\; \begin{array}{l} \gamma : (-\varepsilon, \varepsilon) \to \underset{\underset{GL_n(\mathbb{C})}{\cup\!\shortmid}}{G} \\ \text{is smooth, and} \\ \gamma(0) = I_n \end{array} \right\}$$



e.g. $U(1)$ =



Lemma

$\mathfrak{g} = \mathrm{Lie}(G)$ is a real vector subspace of $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$.

pf

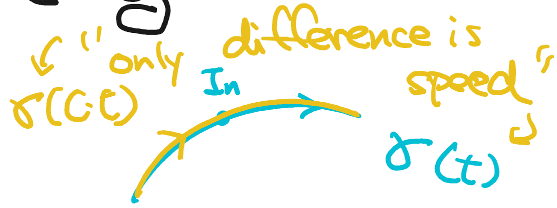
(i) $0 = \big(\underbrace{\text{constant curve at } I_n}_{\gamma(t) = I_n \ \forall t}\big)' = 0$

$\in \mathfrak{g}$

(ii) For $\alpha'(0), \beta'(0) \in \mathfrak{g}$,

$$\alpha'(0)+\beta'(0) = \alpha'(0)\underbrace{\beta(0)}_{\overset{\shortparallel}{I_n}} + \underbrace{\alpha(0)}_{\overset{\shortparallel}{I_n}}\beta'(0)$$

$$= (\underbrace{\alpha\cdot\beta}_{(-\varepsilon,\varepsilon)\to G})'(0) \in \mathfrak{g}$$



(iii) For $c \in \mathbb{R}$, $\gamma'(0) \in \mathfrak{g}$,

$$\left(\gamma(c\cdot t)\right)'\Big|_{t=0} = \gamma'(c\cdot 0)\cdot c$$

$$= c\,\gamma'(0) \in \mathfrak{g} \qquad \#$$

Thm

$\mathfrak{g} = \mathrm{Lie}(G)$ is a real Lie subalgebra of $M_n(\mathbb{C})$ equipped the commutator bracket.

That is, $(\mathfrak{g} \subseteq M_n(\mathbb{C}))$

$$\forall A, B \in \mathfrak{g}, \qquad [A,B] = AB - BA \in \mathfrak{g}$$

pf (sketch)

For $A = \alpha'(0)$, $B = \beta'(0) \in \mathfrak{g}$, we define

$$F : (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon) \longrightarrow G$$

$$F(s,t) = \alpha(s)\cdot\beta(t)\cdot\alpha(s)^{-1}$$

Note:
$F(s,0)$
$= \alpha(s)\cdot I_n \cdot \alpha(s)^{-1}$
$= I_n$

The curve

$$\gamma : (-\varepsilon, \varepsilon) \to \mathfrak{g}, \quad \gamma(s) = \frac{d}{dt}\Big|_{t=0} F(s,t)$$
$$= \underline{\alpha(s)\, \beta'(0)\, \alpha(s)^{-1}} \in \mathfrak{g}$$

is a smooth curve in $\mathfrak{g}$.

Since $\mathfrak{g}$ is a real vector subsp of $M_n(\mathbb{C})$, it is closed in $M_n(\mathbb{C})$, and the limit

$$\lim_{s \to 0} \frac{\gamma(s) - \gamma(0)}{s} = \gamma'(0)$$

$\in \mathfrak{g}$

$-\alpha'(0)$

recall

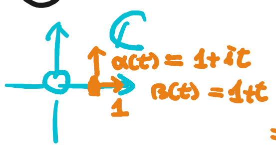
$$= \alpha'(0)\,\beta'(0)\,\underbrace{\alpha(0)^{-1}}_{I_n} + \underbrace{\alpha(0)}_{I_n}\,\beta'(0)\,\boxed{\left(\alpha(s)^{-1}\right)'_{s=0}}$$

$$= \alpha'(0)\,\beta'(0) - \beta'(0)\,\alpha'(0) = AB - BA$$

is still in $\mathfrak{g}$ #

Example

① $\mathrm{Lie}(GL_n(\mathbb{C})) = \mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$

e.g. $GL_1(\mathbb{C}) =$ 

$\Rightarrow \alpha'(0) = i$
$\beta'(0) = 1$
$\Rightarrow i, 1 \in \mathfrak{g}$
$\Rightarrow \mathrm{span}\{i, 1\} \subseteq \mathfrak{g}$
$\mathbb{C} = M_1(\mathbb{C})$

② $\mathrm{Lie}(GL_n(\mathbb{R})) = \mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$

③ $\mathrm{Lie}(SL_n(\mathbb{C})) = \mathfrak{sl}_n(\mathbb{C})$

$$= \{ A \in M_n(\mathbb{C}) \mid \mathrm{tr}(A) = 0 \}$$

why?

Recall: $SL_n(\mathbb{C}) = \{ A \in M_n(\mathbb{C}) \mid \det(A) = 1 \}$

(<u>Lemma</u> $\det(e^A) = e^{\mathrm{tr} A}$)

If $\mathrm{tr}(A) = 0$, then the curve

$$\gamma(t) = e^{tA}$$

maps into $SL_n(\mathbb{C})$ because

$$\det(\gamma(t)) = \det(e^{tA}) = e^{t\,\overset{0}{\overset{\shortparallel}{\mathrm{tr}(A)}}} = 1$$

And

$$\gamma'(0) = A \cdot e^{tA}\Big|_{t=0} = A \in \mathrm{Lie}(SL_n(\mathbb{C}))$$

$\Rightarrow \quad \mathfrak{sl}_n(\mathbb{C}) \subseteq \mathrm{Lie}(SL_n(\mathbb{C}))$

<u>exer</u> : $\supseteq$

④ $\mathrm{Lie}(SL_n(\mathbb{R})) = \mathfrak{sl}_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0 \}$

⑤ $\mathrm{Lie}(U(n)) = \mathfrak{u}(n) = \{ A \in M_n(\mathbb{C}) \mid A + A^* = 0 \}$

pf: skip

⑥ $\mathrm{Lie}(O(n)) = o(n) = \{A \in M_n(\mathbb{R}) \mid A + A^T = 0\}$

$\quad \shortparallel \qquad\qquad\qquad /\!/$

$\mathrm{Lie}(SO(n)) = so(n)$

pf

Recall $O(n) = \{A \in M_n(\mathbb{R}) \mid A^T A = I_n\}$

<u>$\mathrm{Lie}(O(n)) \subseteq o(n)$:</u>

Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow O(n)$, $\gamma(0) = I_n$

$\Rightarrow \quad \gamma(t)^T \cdot \gamma(t) = I_n$

$$\Rightarrow \quad \left.\frac{d}{dt}\right|_{t=0} (\gamma(t)^T \cdot \gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} (I_n) = 0$$

$$\shortparallel$$

$$\boxed{\left.(\gamma(t)^T)'\right|_{t=0}} \cdot \underset{I_n}{\underline{\underline{\gamma(0)}}} + \underset{I_n^T = I_n}{\underline{\underline{\gamma(0)^T}}} \cdot \gamma'(0)$$

$= (\gamma'(0))^T$

$$= \gamma'(0)^T + \gamma'(0) = 0$$

So $\gamma'(0) \in o(n)$

<u>$o(n) \subseteq \mathrm{Lie}(O(n))$</u>

<u>Lemma</u>

$$e^{A^T} = (e^A)^T$$

For $A \in \mathfrak{o}(n)$, i.e. $A^T = -A$, we consider

$$\gamma : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R}), \quad \gamma(t) = e^{tA}$$

$$\Rightarrow \quad \gamma(t)^T = (e^{tA})^T = e^{t \cdot A^T} = e^{-tA}$$

$$= (e^{tA})^{-1} = \gamma(t)^{-1}$$

i.e. $\quad \gamma(t)^T \cdot \gamma(t) = I_n$

i.e. $\quad \gamma(t) \in O(n)$

Furthermore,

$$\gamma'(0) = (e^{tA})' \Big|_{t=0} = A e^{tA} \Big|_{t=0}$$

$$= A \in \mathrm{Lie}(O(n))$$

#

TOPICS IN LINEAR ALGEBRA

HSUAN-YI LIAO

Abstract. Lecture notes for the course "Advanced Linear Algebra" at National Tsing Hua University.

Contents

HW11

2(a). Let $x \in \mathbb{R}$

Show $ev_x : C^\infty(\mathbb{R}) \to \mathbb{R}$, $ev_x(f) = f(x)$

is an alg morphism

pf

$$ev_x(f+g) = f(x) + g(x)$$

$$ev_x(a \cdot f) = a \cdot f(x) \qquad a \in \mathbb{R}$$

$$ev_x(f \cdot g) = (f \cdot g)(x) = f(x) \cdot g(x)$$

$$ev_x(1) = 1(x) = 1$$

#

7. Let $\mathfrak{g}$ be a Lie alg.

Construct a linear $f : \boxed{S\mathfrak{g}} \to U\mathfrak{g}$

s.t. $f|_{S^k\mathfrak{g}} \neq 0 \quad \forall k$

$= T\mathfrak{g} / \langle x \otimes y - y \otimes x \rangle$

sol

Define

$$pbw : S\mathfrak{g} \longrightarrow U\mathfrak{g}$$

by

$$pbw(X_1 \odot \cdots \odot X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)} \cdot X_{\sigma(2)} \cdots X_{\sigma(k)}$$

well-defined #

Details: $\underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k}$ $\quad (X_1, \cdots, X_k)$

multilinear

$$k$$

$$\frac{1}{k!}\sum_{\sigma\in S_k} X_{\sigma(1)}\cdots X_{\sigma(k)}$$

$$\begin{tikzcd}
\mathfrak{g}\otimes\cdots\otimes\mathfrak{g} \arrow[r, "\phi"', "\exists!\ \text{linear}"] \arrow[d] & U\mathfrak{g} \\
S^k\mathfrak{g} \arrow[ur, dashed, "\text{linear pbw}", "\exists!"'] &
\end{tikzcd}$$

$$X_1\otimes\cdots\otimes X_k \longmapsto \frac{1}{k!}\sum_{\sigma\in S_k} X_{\sigma(1)}\cdots X_{\sigma(k)}$$

$$S^k\mathfrak{g} = \mathfrak{g}^{\otimes k} \Big/ \operatorname{Span}\{X_1\otimes\cdots(X_i\otimes X_{i+1} - X_{i+1}\otimes X_i)\cdots X_k\} \xrightarrow{\phi} 0$$

$$\Leftarrow$$