Linear Algebra 12/14

Recall

① $\gamma : I = (-\varepsilon, \varepsilon) \to M_{m\times n}(\mathbb{C})$

is smooth if each component of γ

is smooth $= C^{\infty}$ $\quad I \to \mathbb{R}$

② Exponential map

$$M_n(\mathbb{C}) \xrightarrow{\exp(A) = e^{A}} M_n(\mathbb{C})$$

is defined by

$$\exp(A) = e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^{k}$$

Lemma

The exponential map $\exp : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is well-defined (i.e. the series is convergent).

sketch of pf

Algebraic approach

For $A \in M_n(\mathbb{C})$, $\exists$ invertible $P \in GL_n(\mathbb{C})$

$\exists\, J \leftarrow$ of Jordan canonical form

s.t.

$$A = PJP^{-1}$$

$$A = P J P^{-1}$$

$$\Rightarrow \quad A^k = (PJ\cancel{P^{-1}})\cdot(\cancel{P}J\cancel{P^{-1}})\cancel{\cdot}\cdots\cancel{\cdot}(\cancel{P}JP^{-1})$$

$$= P J^k P^{-1}$$

$$\Rightarrow \quad \exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \frac{1}{k!} (PJ^kP^{-1})$$

$$= P \cdot \left(\sum_{k=0}^{\infty} \frac{J^k}{k!} \right) \cdot P^{-1}$$

$$= P \cdot \exp(J) \cdot P^{-1}$$

And compute $\exp(J)$ directly

e.g.

① $$e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} = ?$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$$

$$e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_2^k \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}} = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{\text{diagonal}} + \overset{\text{niltopent}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

② $e \;=\; ?$

$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^2 = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} = \begin{pmatrix}\lambda^2 & 2\lambda\\ 0 & \lambda^2\end{pmatrix}$$

$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^3 = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}\lambda^2 & 2\lambda\\ 0 & \lambda^2\end{pmatrix} = \begin{pmatrix}\lambda^3 & 3\lambda^2\\ 0 & \lambda^3\end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^k = \begin{pmatrix}\lambda^k & k\lambda^{k-1}\\ 0 & \lambda^k\end{pmatrix}$$

$$\Rightarrow \quad e^{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}} = \sum_{k=0}^{\infty} \frac{1}{k!}\begin{pmatrix}\lambda^k & k\lambda^{k-1}\\ 0 & \lambda^k\end{pmatrix}$$

$$= \begin{pmatrix}\sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k & 1+\lambda+\frac{\lambda^2}{2!}+\cdots\\ 0 & \sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k\end{pmatrix}$$

$$= \begin{pmatrix}e^\lambda & e^\lambda\\ 0 & e^\lambda\end{pmatrix}$$

Analytic approach

Recall

① A normed vector space

$= \text{vector space } V + \underset{\nearrow}{\underbrace{\text{norm}}} \; \|\cdot\|$

(i) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = \vec{0}$

(ii) $\|x+y\| \leq \|x\| + \|y\|$

(iii) $\|c \cdot x\| = |c| \cdot \|x\|$

② A normed vector space is a metric space

③ All the norms $\mathbb{R}^N$ are equivalent

In particular, every norm $\|\cdot\|$ on $\mathbb{R}^N$ is complete

i.e. every Cauchy seq is convergent.

We introduce the operator norm on $M_n(\mathbb{C})$. For $A \in M_n(\mathbb{C})$,

$$\|A\| = \sup_{\substack{\|v\|=1 \\ v \in \mathbb{C}^n}} \underbrace{\|Av\|}_{\text{norm in } \mathbb{C}^n}$$

Lemma

The operator norm is a norm on $M_n(\mathbb{C})$ with the property

$$\|AB\| \leq \|A\| \cdot \|B\|$$

Therefore,

$$\left\| \sum_{k=m}^{n} \frac{A^k}{k!} \right\| \leq \sum_{k=m}^{n} \frac{1}{k!} \overset{\leq (\|A\|)^k}{\|A^k\|}$$

$$\leq \underbrace{\sum_{k=m}^{n} \frac{1}{k!} \|A\|^k}_{\uparrow}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \|A\|^k = e^{\|A\|} \text{ is convergent}$$

$\forall \varepsilon > 0 \quad \exists N$ s.t.

$$\sum_{k=m}^{n} \frac{1}{k!} \|A\|^k < \varepsilon \quad \forall \ m,n \geqslant N$$

$$\Rightarrow \quad \left\| \sum_{k=m}^{n} \frac{1}{k!} A^k \right\| \leqslant \sum_{k=m}^{n} \frac{\|A\|^k}{k!} < \varepsilon \ \forall m,n \geq N$$

So $\left(\sum_{k=1}^{n} \frac{A^k}{k!} \right)_{n=1}^{\infty}$ is a Cauchy seq.

in $(M_n(\mathbb{C}), \underset{\substack{\uparrow \\ \text{operator norm}}}{\|\cdot\|})$

$$\Rightarrow \quad e^A = \sum_{k=1}^{\infty} \frac{A^k}{k!} \quad \text{is convergent in } M_n(\mathbb{C})$$

\#

In fact, this type of Cauchy $\underset{\mathbb{R}^{2n^2}}{\text{seq}}$ $\quad \underset{\shortparallel}{\mathbb{R}^{2n^2}}$

argument also shows $\exp : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is smooth, i.e., ∀ smooth curve $\gamma : I \to M_n(\mathbb{C})$, the composition

$$\exp\circ\gamma : I \xrightarrow{\gamma} M_n(\mathbb{C}) \xrightarrow{\exp} M_n(\mathbb{C})$$

is also smooth.

Prop

If $A, B \in M_n(\mathbb{C})$, $AB = BA$, then

$$\exp(A)\cdot \exp(B) = \exp(A+B).$$

In particular, i.e. $(\exp(A))^{-1} = \exp(-A)$

$$\exp(A)\cdot\exp(-A) = \exp(A-A) = \exp(O_n)$$

$$= I_n = \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}$$

$$\text{LHS} = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)\cdot\left(\sum_{l=0}^{\infty}\frac{B^l}{l!}\right) = \sum_{k,l=0}^{\infty} \frac{1}{k!\,l!} A^k \cdot B^l$$

$$\text{RHS} = \sum_{p=0}^{\infty} \frac{1}{p!}(A+B)^p$$

$$(A+B)^2 = A^2 + AB + \overset{\overset{AB}{\|}}{BA} + B^2$$

To sum up, we have

Thm

The series

$$\exp(A) = e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^{k}$$

defines a smooth map

$$\exp : M_n(\mathbb{C}) \longrightarrow \underline{GL_n(\mathbb{C})}$$

$\uparrow$

$\{ n \times n$ invertible matrixes in $M_n(\mathbb{C}) \}$

Application to differential equation

The curve

$$\mathbb{R} \to M_n(\mathbb{C}), \quad t \mapsto \exp(tA) = e^{tA}$$

has the derivative

$$\left(e^{tA}\right)' = \left(\sum_{k=0}^{\infty} \frac{(tA)^{k}}{k!} \right)'$$

$\cancel{k}\, t^{k-1}$

$\|$

$$= \sum_{k=0}^{\infty} \frac{(t^{k}A^{k})'}{k!} = \sum_{k=0}^{\infty} \frac{\boxed{(t^{k})'}}{\cancel{k!}} A^{k}$$

$(k-1)!$

$$= \boxed{\sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^{k-1}} \cdot A \qquad = e^{tA}$$

$$= \underline{e^{tA} \cdot A} = \underline{A \cdot e^{tA}}$$

Thm

Let $A \in M_n(\mathbb{C})$, and $y : (a,b) \to M_{n\times 1}(\mathbb{C})$ satisfying

$$y' = A \cdot y$$

Then

$$y(t) = e^{tA} \cdot C$$

for some constant $C \in M_{n\times 1}(\mathbb{C})$.

If $y(0) = y_0$, then

$$y(t) = e^{tA} \cdot y_0$$

pf

The equation

$$y' = Ay$$

$$\Rightarrow \quad e^{-tA}(y' - Ay) = 0$$

$$= (e^{-tA})'$$

$$= e^{-tA}\cdot y' \boxed{- e^{-tA} A}\cdot y$$

$$= e^{-tA}\cdot y' + (e^{-tA})'\cdot y$$

$$= \left(e^{-tA}\cdot y\right)' = 0$$

So

$$e^{-tA} y = C$$

for some constant matrix

$C \in M_{n\times n}(\mathbb{C})$.

$$\Rightarrow \quad y = e^{tA}\cdot C \qquad \#$$

<u>Example</u>

$$\underline{y'' + y' - 2y = 0} \qquad ⊛$$

<u>sol</u>

$$\lambda^2 + \lambda - 2 = 0$$

Let $z = y'$

$$\Rightarrow \quad z' = y'' = -y' + 2y = -z + 2y$$

So $\circledast$ is equivalent to

$$\begin{cases} y' = z \\ z' = 2y - z \end{cases}$$

$$\Longleftrightarrow \quad \begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

By Thm,

$$\begin{pmatrix} y \\ z \end{pmatrix} = e^{\underline{\underline{t}}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

for some constants c_1, c_2

Diagonalize $\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$:

$$\det\left(\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right)$$

$$= (-\lambda)(-1-\lambda) - 1 \cdot 2$$

$$, \quad -1$$

$$, \quad 2$$

$$= \lambda^2 + \lambda - 2 = 0 \quad \Rightarrow \lambda = 1, -2$$

$\underline{\lambda=1}$: $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\underline{\lambda=-2}$: $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{\parallel}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & +1 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow \exp\left(t \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \exp\begin{pmatrix} \overset{t}{\cancel{1}} & 0 \\ 0 & \overset{-2t}{\cancel{-2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}}$$

$$\Rightarrow \text{sol} = \begin{pmatrix} y \\ z \end{pmatrix} = \underline{e^{t\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} a e^{t} + b e^{-2t} \\ \ldots \end{pmatrix}$$

So $y = ae^{t} + be^{-2t}$ for some a, b

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Matrix group and its Lie algebra

The symmetries of a space is usually described by a group action, and the theory of groups can be considered as a study of symmetries

Example

$\mathbb{Z}_2 = \{\pm 1\}$ ($\subseteq \mathbb{R}$) acts on $\mathbb{R}$ by multiplication

$A_1 : \mathbb{R} \to \mathbb{R},\ x \mapsto x$

$A_{-1} : \mathbb{R} \to \mathbb{R},\ x \mapsto -x$

Note

- $f : \mathbb{R} \to \mathbb{R}$ is even

  $\Leftrightarrow$ $f(-x) = f(x)$

  $\Leftrightarrow$ f is $\mathbb{Z}_2$ - invariat $\Leftrightarrow$ $f \circ A_y = f \quad \forall y \in \mathbb{Z}_2$

- $f : \mathbb{R} \to \mathbb{R}$ is odd

$\Leftrightarrow \quad f(-x) = -f(x)$

$\Leftrightarrow$ f is $\mathbb{Z}_2$-equivariant

$\Leftrightarrow \quad f \circ A_y = A_y \circ f \quad \forall\, y \in \mathbb{Z}_2$

HW10. 3

(a) $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_c$, $\underline{c = \gcd(m,n)}$

$\Rightarrow m = k \cdot c$

pf

① Note

$$\phi : \overset{\mathbb{Z}/m\mathbb{Z}}{\overset{\shortparallel}{\mathbb{Z}_m}} \times \overset{\mathbb{Z}/n\mathbb{Z}}{\overset{\shortparallel}{\mathbb{Z}_n}} \longrightarrow \overset{\mathbb{Z}/c\mathbb{Z}}{\overset{\shortparallel}{\mathbb{Z}_c}},$$

$x, y, x', y' \in \mathbb{Z}$

$$\phi([x][y]) = [xy]$$

is (i) well-defined because

$$\begin{array}{ccc} \phi([x],[y]) & = & \phi([x+mx'],[y]) \\ \shortparallel & & \shortparallel \\ [xy] & \underset{\text{in } \mathbb{Z}_c}{=} & [xy + \underbrace{m x' \cdot y}_{c \cdot (k \cdot x' \cdot y)}] = [xy] \text{ in } \mathbb{Z}_c \end{array}$$

and

$$\phi([x],[y]) = \phi([x],[y+ny'])$$

(ii) $\mathbb{Z}$-bilinear because

$$\phi(a[x]+b[x'],[y]) = \phi([ax+bx'],[y])$$

$$= [(ax+bx')y] \underset{\text{in } \mathbb{Z}_c}{=} a[xy] + b[x'y]$$

$$= a\phi([x],[y]) + b\phi([x'],[y])$$

and

$$\phi([x], a[y]+b[y']) = a\phi([x],[y]) + b\phi([x],[y'])$$

By the universal property, $\exists!$ $\mathbb{Z}$-linear

$$\tilde{\Phi} : \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \longrightarrow \mathbb{Z}_c$$

s.t. $$\tilde{\Phi}([x]\otimes[y]) = [xy]$$

② The map

$$\tilde{\Psi} : \mathbb{Z}_c \longrightarrow \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$$

$$\tilde{\Psi}([z]) = [z]\otimes[1]$$

is (i) well-defined because

$$\underset{\underset{[z]\otimes 1}{\parallel}}{\tilde{\Psi}([z])} = \underset{\underset{\underset{\parallel}{[z+c\cdot z']\otimes 1}}{\parallel}}{\tilde{\Psi}([z+c\cdot z'])}$$

Note

$c = \gcd(m,n)$

$\Rightarrow \exists\ p, q \in \mathbb{Z}$ s.t.

$c = pm + qn$

$$[z + (pm+qn)\cdot z']\otimes 1$$

in $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$

$0 \otimes 1 = 0$

$$= [z]\otimes 1 + \underbrace{[pm\, z']\otimes 1} + \underbrace{[qn\, z']\otimes 1}$$

$\parallel$

$[m(pz')] = 0$

in $\mathbb{Z}_m$

$\parallel$

$n[qz']\otimes 1$

$= [qz']\otimes \boxed{n\cdot 1}$ $\overset{0}{\parallel}$ in $\mathbb{Z}_n$

$= [qz']\otimes 0 = 0$

$$= [z]\otimes 1 = \tilde{\Psi}([z])$$

$M \otimes N$

Note in $M\otimes_{\mathbb{Z}} N$

$0\otimes y = 0$ because

$$\underline{0\otimes y + \boxed{\cancel{0\otimes y}}} = 0\otimes(y+y) = 0\otimes 2\cdot y$$

$$= 2\cdot 0\otimes y = \boxed{\cancel{0\otimes y}}$$

$$\Rightarrow 0\otimes y = 0$$

(ii) $\tilde{\psi}$ is $\mathbb{Z}$-linear because

$$\tilde{\psi}(a\cdot[z]) = [\bar{a}\cdot z]\otimes 1 = a\cdot([z]\otimes 1)$$

$$= a\cdot\tilde{\psi}([z])$$

In short,

$$\begin{tikzcd} \tilde{\psi} : \mathbb{Z}_c \arrow[r] \arrow[r, bend right, "\tilde{\psi}"'] & \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \arrow[l, bend right, "\tilde{\phi}"'] \end{tikzcd}, \quad \tilde{\psi}([z]) = [z]\otimes 1$$

is $\mathbb{Z}$-linear.

③ $\tilde{\Phi}\circ\tilde{\psi} = id$, because $(\tilde{\Phi}\circ\tilde{\psi})(z) = \tilde{\Phi}(z\otimes 1)$
$= z\cdot 1 = z$

$\tilde{\psi}\circ\tilde{\Phi} = id$, because

$$(\tilde{\psi}\circ\tilde{\Phi})([\bar{x}\otimes y]) = \tilde{\psi}([\bar{x}\cdot y]) = [xy]\otimes 1$$

$$= [x]\otimes y\cdot 1 = [x]\otimes[y]$$

So $\tilde{\Phi}$ is an isomorphism #

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torsion

(b) Let M be a $\mathbb{Z}$-module, i.e.

$$\forall x\in M,\ \exists\ \overset{\neq 0}{n}\in\mathbb{Z} \text{ s.t. } n\cdot x=0$$

Show $M\otimes_{\mathbb{Z}}\mathbb{Q}=0$

pf

Since $M\otimes_{\mathbb{Z}}\mathbb{Q}$ is generated by $\left\{x\otimes\frac{p}{q} \;\middle|\; \begin{array}{l} x\in M \\ \frac{p}{q}\in\mathbb{Q}\end{array}\right\}$

it suffices to show

$$x\otimes\frac{p}{q}=0 \qquad \forall x\in M,\ \forall\frac{p}{q}\in\mathbb{Q}$$

$$\forall x\in M,\ \exists\ \overset{\neq 0}{n}\in\mathbb{Z} \text{ s.t. } n\cdot x=0$$

$$\Rightarrow\quad x\otimes\frac{p}{q}=x\otimes\boxed{n}\,\frac{p}{nq}=\overset{\overset{0}{\shortparallel}}{\underset{\sim}{n\cdot x}}\otimes\frac{p}{nq}$$

$$=0$$

\#

(c) $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}$

pf (sketch)

Check

$\exists!$ $\mathbb{Z}$-linear $\tilde{\Phi}:\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathbb{Q}$, $\tilde{\Phi}(x\otimes y)=xy$

$\tilde{\Psi}:\mathbb{Q}\longrightarrow\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q}$, $\tilde{\Psi}(z)=z\otimes 1$

$\tilde{\Phi}\circ\tilde{\Psi}=id$, $\tilde{\Psi}\circ\tilde{\Phi}=id$

\#

HW10. 9 $\quad \overset{\in}{} V = \Lambda^1 V$

Suppose $\dim V < \infty$. $\alpha \neq \vec{0}$, $\gamma \in \Lambda^k V$.

Show $\quad \alpha \wedge \gamma = 0 \iff \gamma = \alpha \wedge \beta$ for some $\beta \in \Lambda^{k-1} V$

pf

"$\Leftarrow$" is from the basic properties of ΛV

"$\Rightarrow$" Let $e_1, \cdots, e_n$ be a basis for V

s.t. $e_1 = \alpha$ $\quad \{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k\}$ is a basis for $\Lambda^k V$

$$\Rightarrow \overset{\in \Lambda^k V}{\gamma} = \sum_{i_1 < i_2 < \cdots < i_k} a^{i_1 \cdots i_k} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

$$\Rightarrow \alpha \wedge \gamma = e_1 \wedge \gamma \overset{\text{assumption}}{=} 0$$

$$= \sum_{\substack{i_1 < \cdots < i_k \\ i_1 \neq 1}} a^{i_1 \cdots i_k} \underbrace{e_1 \wedge e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k}}_{\text{linearly independent in } \Lambda^{k+1} V}$$

$$\Rightarrow a^{i_1, \cdots i_k} = 0 \quad \forall\ i_1 \neq 1,\ i_1 < \cdots < i_k$$

So $\gamma = \sum_{i_1 < \cdots < i_k} a^{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$

$$= \sum_{i_1 = 1 < i_2 < \cdots i_k} a^{1, i_2, \cdots, i_k} \underbrace{\boxed{e_1}}_{\alpha} \wedge e_{i_2} \wedge \cdots e_{i_k}$$

$$= \alpha \wedge \underbrace{\left(\sum_{i_2 < \cdots < i_k} a^{1, i_2, \cdots i_k} e_{i_2} \wedge \cdots \wedge e_{i_k} \right)}_{= \beta}$$

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