Linear Algebra 12/7

An (associative) algebra has 3 operations
$\wedge$ A over R

R-module (i) addition
(ii) scalar multiplication
(iii) multiplication

ring

s.t.
$$a\cdot(xy) = (a\cdot x)y \qquad \forall\, a\in R,\ x,y\in A$$

Example (Tensor algebra)

$$TV = \bigoplus_{k=0}^{\infty} \overbrace{V\otimes\cdots\otimes V}^{k} \qquad (V^{\otimes 0} = \Bbbk)$$

↖ vector space over $\Bbbk$

with the multiplication characterized by

$$\underbrace{\overset{V^{\otimes k}}{\overset{\cup}{(\vec{v}_1\otimes\cdots\otimes\vec{v}_k)}}}_{} \overset{\text{usually denoted by } \otimes}{\overset{\downarrow}{\bullet}} \underbrace{(\vec{v}_{k+1}\otimes\cdots\otimes\vec{v}_{k+\ell})}_{} \in V^{\otimes \ell}$$

$$= \vec{v}_1\otimes\cdots\otimes\vec{v}_k\otimes\vec{v}_{k+1}\otimes\cdots\otimes\vec{v}_{k+\ell} \in V^{\otimes k+\ell}$$

Recall

If N is a submodule of M, then we have the quotient module M/N ← also a module

If we want to have alg str on $\overset{A}{\overset{\parallel}{M/N}}$,

we shall require the

We shall require the denominator $\hat{N}$ to be an ideal I, i.e.,
(two-sided)

- I is a submodul of A
- $\forall x \in A$, $\forall y \in I$,

$$xy \in I \quad \text{and} \quad yx \in I$$

In this case, one has a natural alg str on A/I.

For a subset $S \subseteq A$, we denote by $<S>$ the ideal generated by S.

Example

① $$SV = TV \big/ \underline{<\{x \otimes y - y \otimes x \mid x, y \in V\}>}$$

$x \odot y - y \odot x = 0$ $\Uparrow$

is called the symmetric alg on V.

The multiplication is denoted by $\odot$

$(x \wedge y) \wedge (z \wedge w) = (z \wedge w) \wedge (x \wedge y)$

② $$\Lambda V = TV \big/ \underline{<\{x \otimes y + y \otimes x \mid x, y \in V\}>}$$

$\Uparrow$ $x \wedge y + y \wedge x = 0$ $x, y \in V$

is called the exterior alg on V

The multiplication is denoted by $\wedge$

Lie algebra

Def

A **Lie algebra** $\mathfrak{g}$ is a vector space $\mathfrak{g}$ together with a bilinear map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

called the **Lie bracket** s.t.

(i) $[x,y] = -[y,x] \qquad \forall x,y \in \mathfrak{g}$

(ii) (Jacobi identity)

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

or equivalently,

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

Example Let A be an alg over $\Bbbk$

① The commutator

$$[x,y] = xy - yx \quad , \qquad x,y \in A$$

is a Lie bracket on A.

$(A, [\,,\,])$ is a Lie alg.

② The derivations of A form a

Lie alg. $\mathrm{Der}(A) = \{ \text{derivations of } A \}$

Here a derivation X is a $\Bbbk$-linear map $X : A \to A$ s.t.

$$X(ab) = X(a)b + aX(b)$$

e.g. $\frac{d}{dx}$ is a derivation of $C^\infty(\mathbb{R})$

For $X, Y \in \mathrm{Der}(A)$.

$$[X,Y] := X \circ Y - Y \circ X$$

Q: Is $[X,Y] \in \mathrm{Der}(A)$?

Check

$$\begin{aligned}
[X,Y](ab) &= X(Y(ab)) - Y(X(ab)) \\
&= X(Y(a)b + aY(b)) \\
&\qquad - Y(X(a)b + aX(b)) \\
&= X(Y(a)b) + X(aY(b)) - Y(X(a)b) - Y(aX(b)) \\
&= X(Y(a))b + \underline{Y(a)\cancel{X(b)}} + \underline{X(a)\cancel{Y(b)}} + aX(Y(b)) \\
&\qquad - Y(X(a))b - \underline{X(a)\cancel{Y(b)}} - \underline{Y(a)\cancel{X(b)}} - aY(X(b)) \\
&= \underbrace{(XY - YX)}_{}\!\!{}^{[X,Y]}(a) \cdot b + a \cdot (XY - YX)^{[X,Y]}(b) .
\end{aligned}$$

#

Conclusion

$(\mathrm{Der}(A), [\,,\,])$ is a Lie alg.

③ Since $M_n(\Bbbk)$ is an alg.,

$(M_n(\Bbbk), \underset{\text{Commutator}}{\underline{[\,,\,]}})$ is a Lie alg.

④ $\mathfrak{sl}_2(\mathbb{C}) = \{ \underset{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\overset{}{A}} \in M_2(\mathbb{C}) \mid \underset{a+d}{\mathrm{tr}(A)} = 0 \}$

whose Lie bracket is

$$[A,B] = AB - BA = 0$$

$\mathrm{tr}[A,B] = \mathrm{tr}(AB) - \underbrace{\mathrm{tr}(BA)}_{} \quad \mathrm{tr}(AB) \overset{\shortparallel}{}$

As a vector space,

$$\dim_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C})) = 3$$

and

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis for $\mathfrak{sl}_2(\mathbb{C})$.

The Lie bracket is determined by

$$[E, F] = \overset{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{\underline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}} - \overset{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}{\underline{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$$

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[H, F] = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} = -2F$$

$$[H,F]=\underline{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}}-\underline{\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}=\begin{pmatrix}0&0\\-2&0\end{pmatrix}=-2F$$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&-1\\0&0\end{pmatrix}$$

$$[H,E]=\underline{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}}-\underline{\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}=\begin{pmatrix}0&2\\0&0\end{pmatrix}=2E$$

Def

Let $\mathfrak{g}$ be a Lie alg. The universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is the quotient alg.

$x\cdot y-y\cdot x=[x,y]$ in $U\mathfrak{g}$ $\forall x,y\in\mathfrak{g}$

$$U\mathfrak{g}=T\mathfrak{g}\Big/\langle\{\underline{x\otimes y-y\otimes x-[x,y]}\mid x,y\in\mathfrak{g}\}\rangle$$

In fact, $U\mathfrak{g}$ is endowed the map

$$\varphi:\ \overset{\cong\,\mathfrak{g}^{\otimes 1}}{\mathfrak{g}}\hookrightarrow T\mathfrak{g}\xrightarrow[]{\text{quotient map}}\!\!\!\!\rightarrow U\mathfrak{g}$$

Prop (Universal property for $U\mathfrak{g}$)

Let $\mathfrak{g}$ be a Lie alg. A be an alg.

If $f:\mathfrak{g}\to A$ is linear with the property

$$f([x,y])=f(x)f(y)-f(y)f(x)\qquad\forall x,y\in\mathfrak{g}$$

(i.e., $f:\mathfrak{g}\to(A,[\,,\,])$ is a Lie alg mor), then

$\exists!$ alg. mor $\tilde{f}:U\mathfrak{g}\to A$ s.t. $\tilde{f}\circ\varphi=f$

$$\begin{tikzcd}\mathfrak{g} \arrow[r, hook, "\varphi"] \arrow[dr] & U\mathfrak{g} \arrow[d, dashed, "\exists !"] \\ & \end{tikzcd}$$

$$\begin{tikzcd} \forall \text{ Lie alg mor} \arrow[dr, "f"'] & \\ & A \arrow[from=1-2, phantom] \end{tikzcd}$$

$$\begin{tikzcd} \arrow[d, dashed, "\exists! \text{ alg mor } \tilde{f}"] \\ A \end{tikzcd}$$

Remark

A representation of a Lie alg $\mathfrak{g}$ is a vector space V together with a Lie alg mor $\phi : \mathfrak{g} \longrightarrow \mathrm{End}(V) = \{f : V \to V \mid f \text{ linear}\}$

A representation of an alg A is a vector space V together with an alg. mor $\Phi : A \longrightarrow \mathrm{End}(V)$.

$a \in A$, $\vec{v} \in V$

$a \cdot \vec{v} = \Phi(a)(\vec{v}) \in V$

In fact, repn of A = A-module V

The universal property of $\mathcal{U}\mathfrak{g}$ implies

$$\begin{matrix} \text{representation} \\ \text{of Lie alg } \mathfrak{g} \end{matrix} = \begin{matrix} \text{representation} \\ \text{of alg } \mathcal{U}\mathfrak{g} \end{matrix}$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, hook] \arrow[dr, "\phi"'] & \mathcal{U}\mathfrak{g} \arrow[d, dashed, "\exists! \Phi"] \\ & \mathrm{End}\, V \end{tikzcd}$$

Matrix

Curves in Complex matrixes

Let $I = (-\varepsilon, \varepsilon)$ be an open interval.

and $\gamma : I \longrightarrow M_{m\times n}(\mathbb{C})$.

We say γ is differentiable (resp. continuous, smooth)

if γ is differentiable (resp. continuous, smooth)

as a map $I \longrightarrow \mathbb{R}^{2mn} \cong M_{m\times n}(\mathbb{C}) \cong \mathbb{C}^{mn}$

More explicitly, the curve

$$\gamma(t) = \begin{pmatrix} a_{11}(t) + i b_{11}(t) & \cdots & a_{1n}(t) + i b_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{m1}(t) + i b_{m1}(t) & \cdots & a_{mn}(t) + i b_{mn}(t) \end{pmatrix}$$

is differentiable (resp. ....) if all the

$a_{jk}(t)$, $b_{jk}(t)$ are differentiable (resp. ...)

If γ is differentiable, its derivative is

$$\gamma'(t) = \begin{pmatrix} a_{11}'(t) + i b_{11}'(t) & \cdots & a_{1n}'(t) + i b_{1n}'(t) \\ \vdots & \ddots & \vdots \\ a_{m1}'(t) + i b_{m1}'(t) & \cdots & a_{mn}'(t) + i b_{mn}'(t) \end{pmatrix}$$

Prop

matrix multiplication

$$\left(\alpha(t)\cdot\beta(t)\right)' = \alpha'(t)\cdot\beta(t) + \alpha(t)\cdot\beta'(t)$$

Therefore, if $\alpha : I \to \underbrace{GL_n(\mathbb{C})}_{n\times n \text{ invertible matrixes}}$ and $\alpha(0) = I_n$,

then

$$\left.\frac{d}{dt}\right|_{t=0} \alpha^{-1}(t) = -\alpha'(0)$$

sketch of pf

1×1 matrx: $M_1(\mathbb{C}) \cong \mathbb{C}$

$$z_1(t) = a_1(t) + i\, b_1(t)$$

$$z_2(t) = a_2(t) + i\, b_2(t)$$

$$\Rightarrow \quad z_1(t)\cdot z_2(t) = a_1a_2 - b_1b_2 + i(a_1b_2 + b_1a_2)$$

$$a_1'\cdot b_2 + a_1\cdot b_2' + b_1'\cdot a_2 + b_1 a_2'$$

|| similarly

$$\Rightarrow \quad (z_1\cdot z_2)' = (a_1a_2 - b_1b_2)' + i\,(a_1b_2 + b_1a_2)'$$

$$= (a_1a_2)' - (b_1b_2)' = a_1'\cdot a_2 + a_1\cdot a_2' - b_1'\cdot b_2 - b_1\cdot b_2'$$

$$= (a_1'a_2 + a_1a_2' - b_1'b_2 - b_1b_2') + i\,(a_1'b_2 + a_1b_2' + b_1'a_2 + b_1a_2')$$

$$z_1'\cdot z_2 + z_1\cdot z_2' = (a_1' + ib_1')(a_2 + ib_2) + (a_1 + ib_1)(a_2' + ib_2')$$

$$= (\underline{a_1' a_2} \overline{- b_1'} b_2) + i(\underline{b_1' a_2} + \boxed{a_1' b_2})$$

$$+ (\underline{a_1 a_2'} \boxed{- b_1 b_2'}) + i(\overline{b_1 a_2'} + \underline{a_1 b_2'})$$

If $\gamma(t) = (z_{jk}(t))_{m\times n}$, $\tilde{\gamma}(t) = (\tilde{z}_{pq}(t))_{n\times r}$

then

$$= z_{jk}' \cdot \tilde{z}_{kq} + z_{jk} \cdot \tilde{z}_{kq}'$$

$$(\gamma \cdot \tilde{\gamma})' = \Big(\sum_k \underline{(z_{jk} \tilde{z}_{kq})'} \Big)_{j,q}$$

$$= \Big(\sum_k z_{jk}' \cdot \tilde{z}_{kq} \Big) + \Big(\sum_k z_{jk} \cdot \tilde{z}_{kq}' \Big)$$

$$= \gamma' \cdot \tilde{\gamma} + \gamma \cdot \tilde{\gamma}'$$

#

If $\alpha : I \to GL_n(\mathbb{C})$, $\alpha(0) = I_n$, then

$$\alpha \cdot \alpha^{-1} = I_n.$$

$$\Rightarrow (\alpha \cdot \alpha^{-1})' = (I_n)' = O_{n\times n}$$

$$\shortparallel$$

$$\alpha' \cdot \alpha^{-1} + \alpha \cdot (\alpha^{-1})'$$

$$\Rightarrow \alpha \cdot (\alpha^{-1})' = -\alpha' \cdot \alpha^{-1}$$

$$\Rightarrow (\alpha^{-1})' = -\alpha^{-1} \cdot \alpha' \cdot \alpha^{-1}$$

$$\Rightarrow (\alpha^{-1})'(0) = -\alpha^{-1}(0) \cdot \alpha'(0) \cdot \alpha^{-1}(0)$$

$$= - I_n^{-1} \cdot \alpha'(0) \cdot I_n^{-1} = - \alpha'(0) \quad \#$$

Prop

If $\gamma : I \to M_n(\mathbb{C})$

$$\gamma(t) = (z_{jk}(t))_{n\times n}$$

is differentiable, then

$$(\det(\gamma(t)))' = \det\begin{pmatrix} z_{11}'(t) & z_{12}(t) \cdots & z_{1n}(t) \\ \vdots & \vdots & \vdots \\ z_{n1}'(t) & \cdots\cdots & z_{nn}(t) \end{pmatrix} +$$

$$\det\begin{pmatrix} z_{11} & z_{12}' & z_{13} & \cdots & z_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{n1} & z_{n2}' & z_{13} & & z_{nn} \end{pmatrix} + \cdots + \det\begin{pmatrix} \cdots & z_{1n}' \\ & \vdots \\ & z_{nn}' \end{pmatrix}$$

pf

$$\det(z_{jk}) = \sum_{\sigma\in S_n} (-1)^\sigma \cdot z_{1\sigma(1)} \cdot z_{2\sigma(2)} \cdots z_{n\sigma(n)}$$

$$\Rightarrow (\det(z_{ik}))' = \sum_{\sigma\in S_n} (-1)^\sigma \underbrace{(z_{1\sigma(1)} \cdot z_{2\sigma(2)} \cdots z_{n\sigma(n)})'}_{\parallel}$$

$$\underline{z_{1\sigma(1)}'} \cdot z_{2\sigma(2)} \cdots z_{n\sigma(n)} + \cdots + z_{1\sigma(1)} \cdots \underline{z_{n-1\sigma(n-1)} \cdot z_{n\sigma(n)}'}$$

$$= \sum_{\sigma\in S_n} (-1)^\sigma \qquad + \sum_{\sigma\in S_n} (-1)^\sigma \qquad + \cdots + \sum_{\sigma\in S_n} (-1)^\sigma \}$$

$$= \det\begin{pmatrix} z_{11}' & \cdots \\ \vdots & \end{pmatrix} + \cdots + \det\begin{pmatrix} & z_{1n}' \end{pmatrix}$$

$$\begin{pmatrix} \vdots & \\ z_{n1}' & \end{pmatrix} \qquad \begin{pmatrix} \cdots & \vdots \\ & z_{nn}' \end{pmatrix}$$

#

Exponential map

The exponential map $\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$? is defined by the power series

$$\exp(A) = e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^{k}$$

which is convergent ? at any matrix $A \in M_n(\mathbb{C})$.

Lemma

The map $\exp : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is well-defined (i.e. convergent).

$\Phi : V^\vee \otimes V \to \mathrm{Hom}(V,V)$, $\Phi(\xi \otimes w) = \langle - | \xi \rangle \cdot w$

$E : V^\vee \otimes V \to \Bbbk$, $E(\xi \otimes v) = \xi(v)$

(b) Show

$$\mathrm{tr} \circ \Phi = E \qquad \circledast$$

pf $\quad V^\vee \otimes V \longrightarrow \Bbbk$

Since both $\mathrm{tr} \circ \Phi$ and E are linear, it suffices to check $\circledast$ on a basis for $V^\vee \otimes V$

Let $\beta = \{e_1, \cdots, e_n\}$ be a basis for V

$\{\hat{e}_1, \cdots, \hat{e}_n\}$ be the dual basis for $V^\vee$

i.e. $\hat{e}_i(e_j) = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$

$\Rightarrow \{ \hat{e}_i \otimes e_j \mid i, j = 1, \cdots, n \}$ is a basis for $V^\vee \otimes V$

(i) $(\mathrm{tr} \circ \Phi)(\hat{e}_i \otimes e_j) = \mathrm{tr}(\underline{\Phi(\hat{e}_i \otimes e_j)})$

$$[\Phi(\hat{e}_i \otimes e_j)]_\beta = \begin{array}{cc} & i\text{-th column} \\ j & \begin{pmatrix} O & \overset{0}{\vdots} & O \\ \cdots & 1 & \\ O & \underset{0}{\vdots} & O \end{pmatrix} \end{array}$$

$V \to V$, $e_k \mapsto \hat{e}_i(e_k) \cdot e_j$

$= \begin{cases} e_j & k = i \end{cases}$

$0 \quad k \neq$

$$\Rightarrow \operatorname{tr}(\Phi(\hat{e}_i \otimes e_j)) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\text{(ii)} \quad E(\hat{e}_i \otimes e_j) = \hat{e}_i(e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

So $\operatorname{tr} \circ \Phi = E$ #