Linear Algebra 12/3

Def

Let M be an R-module. A subset β of M is called a basis for M if

- β generates M, i.e., $\forall x \in M$, $\exists$ $a_1, \cdots, a_n \in R$ and $e_1, \cdots, e_n \in \beta$ s.t.

$$x = a_1 e_1 + \cdots + a_n e_n$$

- β is linearly independent, i.e., $\forall \{e_1, \cdots, e_n\} \subseteq \beta$,

$$a_1 e_1 + \cdots + a_n e_n = 0 \Rightarrow a_1 = \cdots = a_n = 0$$
$$(a_1, \cdots, a_n \in R)$$

An R-module M is free if M has a basis.

Remark

Let β be an arbitrary set. There exists a free R-module $M = R^{(\beta)}$ with basis β.

This module can be constructed by

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$$R^{(\beta)} = \{ \beta \xrightarrow[\text{maps}]{\text{all}} R,\ x \mapsto a_x,\ \text{s.t. } a_x = 0 \text{ except for finitely many } x \in \beta \}$$

$$= \{ \textstyle\sum a_x \cdot x \mid a_x \in R,\ a_x = 0 \text{ except for finitely many } x \in \beta \}$$

$R^{(\beta)}$ is referred as the module freely generated by β.

<u>Prop</u> (Universal property for free module)

Let M be a free module with a basis β, and $i: \beta \hookrightarrow M$ be the inclusion map.

Let N be another R-module and $f: \beta \to N$ be an arbitrary map of sets

Then $\exists!$ R-module morphism

$T_f : M \to N$

s.t.

$$\begin{tikzcd}
\beta \arrow[r, "\forall f"] \arrow[d, hook, "i"'] & N \\
M \arrow[ur, dashed, "\exists!\, T_f"'] &
\end{tikzcd}$$

R-linear

Tensor product of modules

Let R be a commutative ring (with 1_R).

Def

The tensor product $M \otimes_R N$ of R-modules M and N is the quotient module

$$M \otimes_R N = R^{(M \times N)} / K$$

where K is generated by

$$(x+x', y) - (x,y) - (x',y)$$
$$(x, y+y') - (x,y) - (x,y')$$
$$(ax, y) - a(x,y)$$
$$(x, ay) - a(x,y)$$

$(x,y) = 1_R \cdot (x,y) \in R^{(M \times N)}$

for $x, x' \in M$, $y, y' \in N$, $a \in R$. The equivalence class $[(x,y)]$ in $M \otimes_R N$ is denoted $x \otimes y$.

Example

① If V, W are vector spaces over $\Bbbk$, then V and W are $\Bbbk$-modules.

$$V \otimes_{\Bbbk} W = V \otimes W$$

(tensor product as modules) (tensor product as vector spaces)

② $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ are $\mathbb{Z}$-modules (abelian groups).

$$\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 \cong 0$$

because for $x \otimes y \in \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3$,

$$2 \cdot (x \otimes y) = \overset{=0}{\underbrace{(2x)}_{\in \mathbb{Z}_2}} \otimes y = 0 \Rightarrow n \mid 2$$

$$3 \cdot (x \otimes y) = x \otimes \overset{=0}{\underbrace{(3y)}_{\in \mathbb{Z}_3}} = 0 \Rightarrow n \mid 3$$

Let $n =$ order$(x \otimes y)$ = smallest positive integer n s.t. $n \cdot (x \otimes y) = 0$

$\Rightarrow n \mid 2, \quad n \mid 3$

$\Rightarrow n \mid 1 \Rightarrow n = 1$

$$\boxed{\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0}$$

$\Rightarrow n \cdot (x \otimes y) = 1 \cdot (x \otimes y) = x \otimes y = 0$ #

__Def__

Let M_1, M_2, M, N be R-modules. A __bilinear map__ (over R) from $M_1 \times M_2$ to N is characteri by the properties

$$f(rx+sy,\, z) = r f(x,z) + s f(y,z)$$

$$f(x,\, rz+sw) = r f(x,z) + s f(x,w).$$

For $M \otimes N$, one has a canonical bilinear map

$$i : M \times N \longrightarrow M \otimes_R N. \quad i(x,y) = x \otimes y$$

<u>Prop</u> (Universal property for tensor product)

If $f : M_1 \times M_2 \longrightarrow N$ is a bilinear map, then $\exists !$ R-linear map

$$\tilde{f} : M_1 \otimes_R M_2 \longrightarrow N$$

s.t. $f = \tilde{f} \circ i$.

$$\begin{tikzcd}
M_1 \times M_2 \arrow[r, "\forall f"', "\text{bilinear}"] \arrow[d, "i"'] & N \\
M_1 \otimes_R M_2 \arrow[ur, dashed, "\exists !\, \tilde{f}"', "\text{R-linear}"] &
\end{tikzcd}$$

$M_1 \otimes_R M_2$ is uniquely determined by this property up to isomorphism of R-modules

<u>Thm</u>

$$M \otimes_R R \cong M, \qquad R \otimes_R N \cong N$$

$$M \otimes_R N \cong N \otimes_R M$$

$$(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3)$$

$$(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$$

⊛ $\Rightarrow$ there is no ambiguity with

$$M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$$

HW7. 2

$$S^k(V \oplus W) \cong \bigoplus_{i=0}^{k} (S^i V \otimes S^{k-i} W)$$

$$\sum_i (v_1^i + w_1^i) \odot (v_2^i + w_2^i) \odot \cdots \odot (v_k^i + w_k^i)$$

$S^1V \otimes S^1W$

$$k=2: \quad (v_1+w_1) \odot (v_2+w_2) = \underbrace{v_1 \odot v_2}_{\in S^2V \otimes S^0W} + \underbrace{v_1 \odot w_2 + w_1 \odot v_2}_{\in S^1V \otimes S^1W} + \underbrace{w_2 \odot w_2}_{\in S^0V \otimes S^2W}$$

Prop

Let $f: M_1 \to N_1$, $g: M_2 \to N_2$ be R-linear.

Then $\exists!$ R-linear map

$$f \otimes g : M_1 \otimes_R M_2 \longrightarrow N_1 \otimes_R N_2$$

with the property

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y) \quad \forall\, x \in M_1,\ y \in M_2$$

because $M_1 \times M_2 \longrightarrow N_1 \otimes_R N_2$, $(x,y) \mapsto f(x) \otimes g(y)$

is bilinear

Prop (Adjoint functor property)

$$\mathrm{Hom}_R(M_1 \otimes_R M_2, N) \cong \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, N))$$

$(\cong \{R\text{-bilinear map}\})$

Remark

① The detailed proofs can be found in Hungerford's book "Algebra" or Lang's book "Algebra"

② If R is NOT commutative, one still has $M \otimes_R N$ for a right R-module M, and a left R-module N

abelian group

See Hungerford's book for details

(Associative) Algebra

Let R be a commutative ring.

An (associative) algebra over R A is a ring that is also an R-module in such a way that the ring addition and module addition are same, and the multiplications satisfy

$$r\cdot(xy) = (r\cdot x)y = x(r\cdot y) \quad \forall r\in R,\ x,y\in A$$

A commutative algebra is an algebra that is also a commutative ring

An algebra morphism $\phi: A_1 \to A_2$ satisfies

$$\phi(r \cdot x + s \cdot y) = r \cdot \phi(x) + s \cdot \phi(y)$$
$$\phi(xy) = \phi(x)\phi(y) \qquad \forall\, x, y \in A_1$$
$$\phi(1_{A_1}) = 1_{A_2} \qquad r, s \in R$$

Example

① Any ring is a $\mathbb{Z}$-algebra.

② For an R-module M,

(endomorphisms)

$$\mathrm{End}_R(M) = \{\text{R-linear maps } M \to M\}$$

is an R-algebra whose multiplication is given by the composition

③ The ring of $n \times n$ matrixes ($M_n(R)$) with entries in R is an R-algebra.

$$\mathrm{End}_R(R^n) \cong M_n(R)$$

④ The complex numbers form an $\mathbb{R}$-algebra

⑤ The polynomial ring $R[x_1, \dots, x_n]$ is a commutative R-algebra.

In fact, this is the free commutative R-algebra generated by $x_1, \cdots, x_n$.

⑥ Let U be an open subset in $\mathbb{R}^n$.
The space

$$C^\infty(U) = \{ C^\infty \text{ functions } U \to \mathbb{R} \}$$

is an $\mathbb{R}$-algebra

$(f \cdot g)(x) = f(x) \cdot g(x)$ $\quad$ $(r \cdot f)(x) = r f(x)$

$(f+g)(x) = f(x) + g(x)$

<u>Remark</u>

If A and B are R-algebras, then the tensor product $A \otimes_R B$ is also an R-algebra whose product is given by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$$

for $a_1, a_2 \in A$, $b_1, b_2 \in B$.

<u>Def</u> (Tensor algebra)

Let V be a vector space over a field $\mathbb{k}$.

The tensor algebra TV is

$$TV = \bigoplus_{k=0}^{\infty} T^k V.$$

k times

$$T^0V = \Bbbk, \quad T^kV = \overbrace{V \otimes_{\Bbbk} V \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} V}$$

together with the multiplication $\otimes$.

That is,

$$\underbrace{(\vec{v}_1 \otimes \cdots \otimes \vec{v}_k)}_{\in\, T^kV} \overset{\text{product in } TV}{\otimes} \underbrace{(\vec{v}_{k+1} \otimes \cdots \otimes \vec{v}_{k+l})}_{\in\, T^lV} = \vec{v}_1 \otimes \cdots \otimes \vec{v}_k \otimes \vec{v}_{k+1} \otimes \cdots \otimes \vec{v}_{k+l}$$

e.g.

$$(\vec{v}_1 \otimes \vec{v}_2 + \vec{v}_3 \otimes \vec{v}_1 \otimes \vec{v}_2) \otimes \vec{v}_4 = \vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_4 + \vec{v}_3 \otimes \vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3$$

The tensor algebra TV is an algebra over $\Bbbk$ and is also called the free algebra on V because of the following property

Prop (Universal property for tensor algebra)

Any $\Bbbk$-linear map $f : V \to A$ from a $\Bbbk$-vector space V to a $\Bbbk$-algebra A can be uniquely extended to an algebra morphism $\tilde{f} : TV \to A$

$$\begin{tikzcd}
\overset{T^1V}{=} V \arrow[r, hook] \arrow[dr, "\forall \text{ linear } f"'] & TV \overset{=\bigoplus_{k=0} T^kV}{} \arrow[d, dashed, "\exists!\, \tilde{f} \text{ alg mor}"] \\
 & A
\end{tikzcd}$$

Remark

The $\Bbbk$-vector space V in the construction of TV can replaced by an R-module M.

$$TM = \bigoplus_{k=0}^{\infty} \overbrace{M \otimes_R \cdots \otimes_R M}^{k}$$

One also has an analogous universal property.

Quotient algebra

Def

Given a ring R, a (two-sided) ideal I in R is a subset of R s.t.

(i) I is a subgroup of $(R,+)$

(ii) $\forall r \in R$, $x \in I$, one has

$$rx \in I, \quad xr \in I$$

not necessary if R is commutative

The quotient ring R/I is the set of equivalence classes

$$[r] = r + I = \{ r + x \mid x \in I \}$$

together with the operations

(i) $[r] + [s] = [r+s]$

(ii) $[r] \cdot [s] = [r \cdot s]$

Lemma

Lemma

$(R/I, +, \cdot)$ is a ring, called a quotient ring.

Def

A (two-sided) ideal in an R-algebra A is a (two-sided) ideal I in the underlying ring $(A, +, \cdot)$ which is closed under scalar multiplication.

The quotient algebra A/I is the algebra whose underlying ring is the quotient ring A/I and whose module structure is given by

$$r \cdot [x] = [r \cdot x]$$

exer

① A/I is an algebra

② Show that any intersection of ideals is still an ideal.

Def

Let S be a subset of an R-algebra A

The two-sided ideal generated by S is

$$\langle S \rangle = \bigcap \{ I \mid I \supseteq S,\ I \text{ is a two-sided ideal in } A \}$$

which is the smallest two-sided ideal containing S.

Example

$A = \Bbbk[x]$, $S = x^2$

$$\langle S \rangle = \langle x^2 \rangle = \{ f(x) \cdot x^2 \mid f(x) \in \Bbbk[x] \}$$

$$= \{ a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots a_n x^n \}$$

subspace

ideal because $(b_0 + b_1 x^1 + \cdots)(a_2 x^2 + \cdots)$

$= b_0 a_2 x^2 + (b_1 a_2 + b_0 a_3) x^3 + \cdots$

$\in \langle S \rangle$

$\Rightarrow$ we have the quotient algebra

$$\Bbbk[x] / \langle x^2 \rangle = \{ [a_0 + a_1 x^1] \} \cong \Bbbk^2$$

as vector space

with the operations

$$c_1 [a_0 + a_1 x] + c_2 [b_0 + b_1 x] = [(c_1 a_0 + c_2 b_0) + (c_1 a_1 + c_2 b_1) x]$$

$$[a_0 + a_1 x]\cdot[b_0 + b_1 x] = [\,a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \underbrace{a_1 b_1 x^2}_{\in \langle x^2 \rangle}\,]$$

$$= [\,a_0 b_0 + (a_0 b_1 + a_1 b_0)x\,]$$

<u>Def</u>

The symmetric algebra SV on a vector space V is the quotient algebra

$$SV = TV \Big/ \underbrace{\langle \{ x\otimes y - y \otimes x \mid x, y \in V \} \rangle}_{\Rightarrow\ x \odot y = y \odot x}$$

Notation:

$x \odot y = [x \otimes y]$

$\odot$ = the product

We denote by $S^k V$ the image $T^k V$ under the quotient map.

<u>Prop</u> (Universal property for symmetric algebra)

Let A be a commutative algebra over $\Bbbk$.

For every linear map

$$f : V \longrightarrow A$$

$\exists!$ algebra morphism

$$\tilde{f} : SV \longrightarrow A$$

s.t. $f = \tilde{f} \circ i$, where $i : V \cong S^1 V \hookrightarrow SV$.

$$\begin{tikzcd} V \arrow[r, "\forall f"', "\text{linear}"] \arrow[d, hook] & A \ (\text{commutative}) \\ SV \arrow[ur, dashed, "\exists!\, \tilde{f}"', "\text{alg mor.}"] & \end{tikzcd}$$

<u>exer</u>

$$\Bbbk[x_1, \dots, x_n] \cong S(\Bbbk^n)$$

<u>Def</u>

The <u>exterior algebra</u> ΛV on a vector space V is the quotient algebra

$x \wedge y = -y \wedge x$
$x \wedge y + y \wedge x = 0$
$\Rightarrow$

$$\Lambda V = TV \big/ \langle \{ x\otimes y + y \otimes x \mid x, y \in V \} \rangle$$

We denote by $\Lambda^k V$ the image of $T^k V$ under the quotient. and $x \wedge y = [x \otimes y]$

<u>Prop</u> (Universal property for exterior algebra)

Let A be an algebra over $\Bbbk$. For every linear map $f: V \to A$ with the property

$$f(x)f(y) + f(y)f(x) = 0,$$

$\exists!$ alg. mor. $\tilde{f}: \Lambda V \to A$ s.t. $f = \tilde{f} \circ i$, where $i: V = \Lambda^1 V \hookrightarrow \Lambda V$.

$$\begin{tikzcd} V \arrow[r, "f"', "\forall \text{ linear}"] \arrow[d, hook, "i"'] & A \\ \Lambda V \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

$f(x)f(y) + f(y)f(x) = 0$

alg. mor.

<u>Remark</u>

The $\Bbbk$-vector space V in the constructions of SV, ΛV can be replaced by an R-module M. One also has analogous

$\Bbbk$-module M. One also has analogous universal properties.

Lie algebra

← an important type of nonassociative algebras

Suppose $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ for convenience.

Def

A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket, satisfying the following properties

(i) $[x,y] = -[y,x]$

(ii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Jacobi identity

Note: associativity:

$$[[x,y],z] = [x,[y,z]]$$

is NOT true for most Lie algebras $\mathfrak{g}$!!