# Linear Algebra 11/23

## Remark

Let $U$ be an open subset of $\mathbb{R}^n$.

$$\begin{tikzcd} U \times \Lambda^p(\mathbb{R}^n)^\vee \arrow[d] & = \Lambda^p(\text{cotangent bundle}) \\ U \arrow[u, bend right] & \end{tikzcd}$$

Then

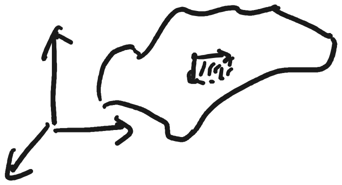
$$\Omega^p(U) = \{\text{differential } p\text{-forms on } U\}$$

$$= C^\infty(U) \otimes_{\mathbb{R}} \Lambda^p(\mathbb{R}^n)^\vee = \left\{ \sum \underset{\overset{\in}{C^\infty(U)}}{f^{i_1 \cdots i_p}} \otimes \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} \right\}$$

If $p = n$,

span{det}

$$\Omega^n(U) = C^\infty(U) \otimes_{\mathbb{R}} \underline{\Lambda^n(\mathbb{R}^n)^\vee}$$



## Bilinear forms

$V$: vector space over $\mathbb{k}$, $\dim V < \infty$

### Def

A bilinear form $B$ is a bilinear map $B : V \times V \to \mathbb{k}$.

A bilinear form $B$ is nondegenerate if $B(\vec{v},\vec{w})=0 \ \forall \vec{w}\in V \Rightarrow \vec{v}=0$

$\Leftrightarrow B^{\#}(\vec{v}) =$ zero map

$\left( \Leftrightarrow \quad B^{\#}: V \longrightarrow V^{\vee}, \quad \vec{v} \mapsto B(\vec{v},-) \text{ is iso} \right)$

We say $B$ is symmetric if

$$B(\vec{v},\vec{w}) = B(\vec{w},\vec{v}) \quad \forall \vec{v},\vec{w}\in V.$$

If $\mathbb{k}=\mathbb{C}$, we say $B: V\times V\to\mathbb{C}$ is a sesquilinear form if

- $B(\vec{u}+\vec{v},\vec{w}) = B(\vec{u},\vec{w}) + B(\vec{v},\vec{w})$
  $B(\vec{u},\vec{v}+\vec{w}) = B(\vec{u},\vec{v}) + B(\vec{u},\vec{w})$
- $B(\lambda\vec{u},\vec{v}) = \bar{\lambda}\, B(\vec{u},\vec{v})$
  $B(\vec{u},\lambda\vec{v}) = \lambda\, B(\vec{u},\vec{v})$

Example

① $\mathbb{k}\times\mathbb{k} \to \mathbb{k}, \quad (a,b)\mapsto ab$
is a bilinear form
(symmetric, nondegenerate)

② Inner products on real vector spaces

are symmetric nondegenerate bilinear forms

Recall

An inner product

$$\langle\ ,\ \rangle : V \times V \to \mathbb{R}$$

is a symmetric, bilinear map s.t.

$$\langle \vec{v}, \vec{v} \rangle > 0 \qquad \forall\ \vec{v} \neq 0.$$

[positive definite

$\Rightarrow$ nondegenerate

$\because\ \langle \vec{v}, \vec{w} \rangle = 0\ \ \forall\ \vec{w}$

$\Rightarrow\ \langle \vec{v}, \vec{v} \rangle = 0$

$\Rightarrow\ \vec{v} = 0$

③ $M_{m\times n}(\mathbb{k}) \times M_{m\times n}(\mathbb{k}) \to \mathbb{k}$

$$(A, B) \mapsto tr(A \cdot B^T)$$

is a bilinear form.

Def

A bilinear form $\omega : V \times V \to \mathbb{k}$

is skew-symmetric (or antisymmetric) if $\omega(\vec{v},\vec{w}) = -\omega(\vec{w},\vec{v}) \quad \forall \vec{v},\vec{w} \in V$

A symplectic bilinear form is a nondegenerate skew-symmetric bilinear

Example

The bilinear form $\omega : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$\omega\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = x_1 y_2 - x_2 y_1$$

$\leftrightarrow$ Hamiltonian system in classical mechanics

is a symplectic bilinear form.

skew-symmetric:

$$\omega(\vec{y},\vec{x}) = y_1 x_2 - y_2 x_1 = -(x_1 y_2 - x_2 y_1)$$

$$= -\omega(\vec{x},\vec{y})$$

Remark $(\dim V = n < \infty)$

① $\{$symmetric bilinear forms on $V\}$

$\cong S^2 V^\vee$

$\Rightarrow \dim = \binom{n+1}{2}$

Recall

$S^2 V^\vee \cong \mathrm{Hom}(S^2 V, \mathbb{k})$

$= \{$symmetric bilinear maps

$T: V \times V \to \mathbb{k}$

② $\{$ skew-symmetric bilinear forms on $V\}$

$$\cong \Lambda^2 V^\vee$$

$$\Rightarrow \dim = \binom{n}{2}$$

③ $\{$ nondegerate bilinear forms $\}$

is NOT a vector space

<u>Prop</u> (Coordinate representation)

Let $\{e_1, \cdots, e_n\}$ be a basis for $V$.

A bilinear linear form $V$ is uniquely determined by

$$B_{ij} := B(e_i, e_j) \in \mathbb{k}$$

$\forall\ i, j = 1, \cdots, n$

Furthermore, if $\vec{v} = \sum_{i=1}^{n} a_i e_i$, $\vec{w} = \sum_{j=1}^{n} b_j e_j$

then

$$B(v, w) = \sum_{i,j=1}^{n} a_i b_j B_{ij} \qquad ⊛$$

$$= \left( \quad \right) \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \end{pmatrix}$$

$$= (a_1 \cdots a_n)_{1\times n} \begin{pmatrix} & \vdots & \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}_{n\times n} \begin{pmatrix} \vdots \\ b_n \end{pmatrix}_{n\times 1}$$

**Prop**

① $B$ is nondegenerate $\Leftrightarrow$ $(B_{ij})$ is invertible

② $B$ is symmetric $\Leftrightarrow$ $(B_{ij})$ is symmetric, i.e.,

$$(B_{ij})^T = (B_{ij})$$

③ $B$ is skew-symmetric $\Leftrightarrow$ $(B_{ij})$ is skew-symmetric

i.e. $$(B_{ij})^T + (B_{ij}) = 0$$

pf (sketch)

①: $(B_{ij})$ = matrix representation of

$$B^{\#} : V \to V^{\vee}, \quad \vec{v} \mapsto B(\vec{v}, -)$$

② & ③: Use ⊛

$$B(\vec{v},\vec{w})^T = B(\vec{v},\vec{w}) \begin{cases} \overset{②}{=} B(\vec{w},\vec{v}) \\ \overset{③}{=} -B(\vec{w},\vec{v}) \end{cases}$$

$$\|$$

$$\left( (a_1 \cdots a_n)(B_{ij})\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right)^T = (b_1 \cdots b_n)\boxed{(B_{ij})^T}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$= B^T(\vec{w},\vec{v})$$

②: $B(\vec{v},\vec{w}) = B(\vec{w},\vec{v}) = (b_1 \cdots b_n)(B_{ij})\begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix}$

$\Rightarrow (B_{ij}) = (B_{ij})^T$

## Module

Recall a ring is a set $R$ equipped with two binary operations $+$ and $\cdot$ satisfying

① $(R,+)$ is a "abelian group" i.e.

(a) $(a+b)+c = a+(b+c)$

(b) $a+b = b+c$

(c) $\exists\ 0 \in R$ s.t. $a+0=a$ $\forall a \in R$

(d) $\forall a \in R$, $\exists -a \in R$ s.t.

$$a+(-a)=0$$

② $(R,\cdot)$ is a "monoid", i.e.

(a) $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ (eg. Hungerford)

(b) $\exists\ 1_R \in R$ s.t. ← Some people don't require this

$$a\cdot 1_R = 1_R\cdot a = a \quad \forall a \in R$$

③ "$\bullet$" is distributive w.r.t. "$+$" i.e.

(a) $a \cdot (b+c) = a \cdot b + a \cdot c$

(b) $(b+c) \cdot a = b \cdot a + c \cdot a$

**Example** Following are rings

① Any fields.

② $\mathbb{Z} = \{\text{integers}\}$

③ $M_n(\mathbb{k}) = \{n \times n \text{ matrixes}\}$

④ $C^\infty(\mathbb{R}^n) = \{\text{smooth } \overset{\mathbb{R}^n \to \mathbb{R}}{\text{functions}}\}$

$(f \cdot g)(x) = f(x) \cdot g(x)$

⑤ $\mathbb{k}[x_1, \dots, x_n]$

$\boxed{R = \text{ring}}$

**Def**

A (left) $R$-module $M$ consists of an abelian group $(M, \overset{= +_M}{+})$ and a scalar multiplication

$$\bullet : R \times M \longrightarrow M$$

s.t. $\forall r, s \in R$, $\forall x, y \in M$

① $r \cdot (x +_M y) = r \cdot x +_M r \cdot y$

② $(r +_R s)\cdot x = r\cdot x +_M s\cdot x$

③ $(rs)\cdot x = r\cdot(s\cdot x)$

④ $1_R\cdot x = x$ ←

(Hugerford)
Some authors
don't require ④
They say "unital
module"
if ④ is satisfied

A right R-module is defined similarly

with $\bullet : M\times R \to R$

( Main difference:
$x\cdot(rs) = (x\cdot r)\cdot s$ )

Example

① Any $\Bbbk$-vector sps are $\Bbbk$-module

② Any abelian gp is a $\mathbb{Z}$-module

Let $(A,+)$ be an abelian gp.

$n\in\mathbb{Z}$, $x\in A$

If $n>0$,

$$n\cdot x := \overbrace{x+x+\cdots+x}^{n\text{ times}}$$

If $n=0$,

$$0\cdot x := 0$$

If $n<0$,

$$\overbrace{\qquad\qquad}^{-n\text{ times}}$$

$$n \cdot x := -(x + x + \cdots + x)$$

③ $\overbrace{R \times R \times \cdots \times R}^{n}$ is an R-module

$$a \cdot (x_1, \cdots, x_n) = (ax_1, \cdots, ax_n)$$

④ $M_n(R)$ is an R-module.

⑤ {derivations of $C^\infty(\mathbb{R}^n)$}

$$= \left\{ X : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \text{ is } \mathbb{R}\text{-linear and } X(fg) = X(f)\cdot g + f \cdot X(g) \right\}$$

eg $X = \frac{\partial}{\partial x_1}$

is a $C^\infty(\mathbb{R}^n)$-module.

"For $a \in C^\infty(\mathbb{R}^n)$,

$$(a \cdot X)(f) = \overset{C^\infty(\mathbb{R}^n)}{\underset{\downarrow}{}}\underline{a} \cdot (\overset{C^\infty(\mathbb{R}^n)}{\underset{\downarrow}{}}\underline{X(f)}) \quad \forall f \in C^\infty(\mathbb{R}^n)$$

<u>Remark</u> (Left R-module v.s. right R-module)

Let M be a left R-module

N be a right R-module

Let

$L_a(x) := a \cdot x$ $\qquad$ $a, b \in R$

$R_b(y) := y \cdot b$ $\qquad$ $x \in M$, $y \in N$

Th

Then

(i) $(L_a \circ L_b)(x) = a\cdot(L_b(x)) = a\cdot(b\cdot x)$

$$= (ab)\cdot x = L_{ab}(x)$$

i.e. $L_a \circ L_b = L_{ab}$

(ii) $(R_a \circ R_b)(y) = (R_b(y))\cdot a = (y\cdot b)\cdot a$

$$= y\cdot(ba) = R_{ba}(y)$$

i.e. $R_a \circ R_b = R_{ba}$

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$$S^3(V\oplus W) = \Big\{ \sum x_1\odot x_2\odot x_3 \;\Big|\; \overset{\vec{v}_1+\vec{w}_1}{\overset{\shortparallel}{x_1}},\ \overset{\vec{v}_2+\vec{w}_2}{\overset{\shortparallel}{x_2}},\ \underset{\vec{v}_3+\vec{w}_3}{\underset{\shortparallel}{x_3}} \in V\oplus W \Big\}$$

$$\shortparallel$$

$$(\vec{v}_1+\vec{w}_1)\odot(\vec{v}_2+\vec{w}_2)\odot(\vec{v}_3+\vec{w}_3)$$

$$= \underset{\substack{\rotatebox{90}{$\in$} \\ S^3V}}{\underline{\vec{v}_1\odot\vec{v}_2\odot\vec{v}_3}} + \underset{\substack{\rotatebox{90}{$\in$} \\ S^2V\otimes W}}{\vec{v}_1\odot\vec{v}_2\odot\vec{w}_3} + \cdots$$

HW7.4.(c)

because

$1\leftrightarrow 2$ $\quad v_2\otimes v_1+v_1\otimes v_2$

$$A = \overset{\shortparallel}{\underset{\leftrightarrow}{v_1\otimes v_2} + \underset{\leftrightarrow}{v_2\otimes v_1}} \in Sym^2(V), \quad B = v_3 \in Sym^1(V)$$

But $A\otimes B \notin Sym^3(V)$

Find $C \in Sym^3(V)$ s.t. $\pi(C) = \pi(A)\odot\pi(B)$

Recall:

$$S^k V = \overbrace{V\otimes\cdots\otimes V}^{k} \Big/ \qquad \xleftarrow{\pi} V\otimes\cdots\otimes V$$

$$\xrightarrow[\cong]{\boxed{\text{sym}}} \quad \{\text{symmetric } k\text{-tensors}\} = \mathrm{Sym}^k V \subseteq V^{\otimes k}$$

$$\mathrm{Sym}(\vec{v}_1 \odot \cdots \odot \vec{v}_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \vec{v}_{\sigma(1)} \otimes \cdots \otimes \vec{v}_{\sigma(k)} \in$$

$$C = \overset{\text{Symmetrization}}{(v_1 \otimes v_2 + v_2 \otimes v_1) \otimes v_3}$$

$$= \text{Symmetrization}\left( \underline{v_1 \otimes v_2 \otimes v_3} + v_2 \otimes v_1 \otimes v_3 \right)$$

$$= \underline{\frac{1}{6}}\left(v_1 \otimes v_2 \otimes v_3 + \underline{v_1 \otimes v_3 \otimes v_2 + v_2 \otimes v_1 \otimes v_3 + v_2 \otimes v_3 \otimes v_1} \right.$$

$$\left. + \underline{v_3 \otimes v_1 \otimes v_2 \otimes v_3 \otimes v_2 \otimes v_1} \right)$$

$$+ \frac{1}{6}\left( \quad \cdots\cdots \quad \right) \in \mathrm{Sym}^3(V)$$

$$\pi(C) = \frac{1}{6}\pi(\cdots) + \frac{1}{6}\pi(\cdots)$$

$$= \pi(v_1 \otimes v_2 \otimes v_3) + \pi(v_2 \otimes v_1 \otimes v_3)$$

$$= 2\, v_1 \odot v_2 \odot v_3 = \pi(A) \odot \pi(B)$$

$$\begin{tikzcd}
\mathrm{Sym}^k(V) \times \mathrm{Sym}^\ell(V) \arrow[r, "\tilde{\odot}"] \arrow[d, shift left, "\pi"] & \mathrm{Sym}^{k+\ell}(V) \\
S^k(V) \times S^\ell(V) \arrow[r, "\odot"] \arrow[u, shift left, "\cong\ \mathrm{sym}"] \arrow[u, shift right, "\pi"'] & S^{k+\ell}(V) \arrow[u, "\mathrm{sym}"]
\end{tikzcd}$$

$S(V) \times S(W)$

$$A \tilde{\odot} B = \mathrm{Sym}\left(\pi(A) \odot \pi(B)\right) \qquad \#$$

<u>Example</u>

$M_n(\mathbb{k})$ is a left $M_n(\mathbb{k})$-module : $A \cdot X = \overset{\text{matrix} \,\cdot}{\overset{\downarrow}{AX}}$

is also a right $M_n(\mathbb{k})$-module : $X \cdot A = XA$

$$(L_A \circ L_B)(X) = ABX = L_{AB}(X)$$

$$(R_A \circ R_B)(X) = XBA = R_{BA}(X) \neq \underset{\underset{XAB}{\shortparallel}}{R_{AB}(X)}$$

<u>Remark</u> (Relation between left R-mod and right R-mod )

Let $R^{op}$ be the ring with

underlying set $= R$

and multiplication $\overset{\cdot \text{ in } R}{\swarrow}$

$$a \,\square\, b := b \cdot a$$

e.g. $M_n(\mathbb{k})^{op}$ $\quad A \,\square\, B = BA$

One has the bijection

left $\quad$ right

$$\begin{Bmatrix} \text{left} \\ R\text{-module} \end{Bmatrix} \longleftrightarrow \{ R^{op}\text{-module} \}$$

$$\underset{\in R}{a} \cdot x \longleftrightarrow x \square \underset{\uparrow R^{op}}{\underline{a^{op}}}$$

More precisely, let $M$ be a left $R$-mod.

Define a right $R^{op}$-module structure on $M$ by

$$x \square a^{op} := a \cdot x$$

This a right $R^{op}$-module because

$$\begin{aligned} \underline{(x \square a^{op}) \square b^{op}} &= (a \cdot x) \square b^{op} \\ &= b \cdot (a \cdot x) \overset{\text{left } R\text{-mod axiom}}{=} (b \cdot a) \cdot x \\ &\overset{\text{def of } R^{op}}{=} (a^{op} \square b^{op}) \cdot x \\ &= \underline{x \square (a^{op} \square b^{op})} \end{aligned}$$

$\Rightarrow$ it satisfies axiom of right $R^{op}$-mod

**Def**

Let $M$ and $N$ be (left) $R$-modules.

A map $f : M \to N$ is called a homomorphism of $R$-modules or $R$-linear map if

$$f'(a\cdot x+b\cdot y)=a\cdot f(x)+b\cdot f(y)$$

$\forall a,b\in R \quad \forall x,y\in M.$

isomorphism = 1-1, onto, R-linear map

$\ker(f)=f^{-1}(0)$, $\operatorname{im}(f)=f(M)$

A submodule of M is a subset of M which is also an R-module under the operations of M.

Let N be a submodule of M.

$$M/N := M/\sim$$

where

$$x\sim y \iff x-y\in N.$$

The set $M/N$ is equipped an R-module structure

$$[x]+[y]=[x+y]$$
$$a\cdot[x]=[a\cdot x]$$

Lemma $(M/N, +, \cdot)$ is an R-module

called a quotient modul

— called a quotient module

Direct sums and direct products also can be defined similarly as vector spaces.

Remark (Difference between modules and vector spaces)

Main difference is an $R$-module does NOT necessarily have a basis.

e.g.

$M = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}$-module

$= \{0, 1\}$

$n \cdot 0 = 0 \quad \forall n \in \mathbb{Z}$

$$n \cdot 1 = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Recall

$B = \{e_1, \ldots, e_n\}$ is a basis

$\Leftrightarrow \forall x \in M, \ \exists! \ c_1, \ldots, c_n \in R$

s.t. $x = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$

In the case of $M = \mathbb{Z}_2$, $\nexists$ such $B$

Suppose $B$ is a basis for $\mathbb{Z}_2$.

$$\Rightarrow \overset{\text{impossible}}{\exists!}\; c_1, \dots, c_n \in \mathbb{Z}$$

$$1 = c_1x_1 + \cdots + c_nx_n$$

But

$$1 = \overbrace{\underset{\neq\, c_1}{(c_1+2)}\,x_1}^{=\, c_1x_1 + 2x_1 \,=\, c_1x_1} + c_2x_2 + \cdots + c_nx_n$$

$\rightarrow\leftarrow$ to "!" $\#$

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HW6.5

(a) Show $\exists!$ linear map $E : V^\vee \otimes V \to \Bbbk$ s.t.

$$E(\xi \otimes v) = \xi(v)$$

pf

Note the map

$$\begin{aligned} B : V^\vee \times V &\longrightarrow \Bbbk \\ (\xi, v) &\longmapsto \xi(v) \end{aligned}$$

is a bilinear map, because

$$\begin{aligned} B(a_1\xi_1 + a_2\xi_2, v) &= (a_1\xi_1 + a_2\xi_2)(v) \\ &= a_1\xi_1(v) + a_2\xi_2(v) \end{aligned}$$

$$= a_1 B(\xi_1, v) + a_2 B(\xi_2, v)$$

and

$$B(\xi, b_1 v_1 + b_2 v_2) = \xi(b_1 v_1 + b_2 v_2)$$

$$= b_1 \xi(v_1) + b_2 \xi(v_2)$$

$$= b_1 B(\xi, v_1) + b_2 B(\xi, v_2)$$

By the universal property of $\otimes$, $\exists!$ linear

$$E : V^\vee \otimes V \to \Bbbk$$

s.t. $E(\xi \otimes v) = B(\xi, v) = \xi(v)$ #

(b) Show $\mathrm{tr} \circ \Phi = E$, where

$$\Phi : V^\vee \otimes V \to \mathrm{Hom}(V, V),$$

$$\Phi(\xi \otimes v)(w) = \xi(w) \cdot v.$$

Let $\{e_1, \cdots, e_n\}$ be a basis for $V$

$\{\hat{e}_1, \cdots, \hat{e}_n\}$ be the dual basis for $V^\vee$.

$\Rightarrow$ $\{\hat{e}_i \otimes e_j \mid i, j = 1, \cdots, n\}$ is a basis for $V^\vee \otimes V$

So

$$\mathrm{tr} \circ \Phi = E$$

$$\mathrm{tr}\circ\overline{\Phi} = E$$

$\Longleftrightarrow$

$$(\mathrm{tr}\circ\overline{\Phi})(\hat{e}_i\otimes e_j) = E(\hat{e}_i\otimes e_j)$$

$$\forall\, i,j = 1,\dots,n$$

<u>LHS:</u>

Since $\overline{\Phi}(\hat{e}_i\otimes e_j)(e_k) = \hat{e}_i(e_k)\cdot e_j = \begin{cases} e_j & k=i \\ 0 & k\neq 0 \end{cases}$

$$\Rightarrow \left(\overline{\Phi}(\hat{e}_i\otimes e_j)\right) = \begin{array}{cc} & \begin{array}{ccc} & i & \end{array} \\ j & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array}$$

$$\Rightarrow \mathrm{tr}\left(\overline{\Phi}(\hat{e}_i\otimes e_j)\right) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

<u>RHS:</u>

$$E(\hat{e}_i\otimes e_j) = \hat{e}_i(e_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \quad \#$$