Linear Algebra 11/9

Recall

$\vec{v}_i \odot \vec{v}_{i+1} = \vec{v}_{i+1} \odot \vec{v}_i$

$$S^k V = V^{\otimes k} \Big/ \operatorname{span}\{ v_1 \otimes \cdots (v_i \otimes v_{i+1} - v_{i+1} \otimes v_i) \otimes \cdots v_k \}$$

$\vec{v}_i \wedge \vec{v}_{i+1} = - \vec{v}_{i+1} \wedge \vec{v}_i$

$$\Lambda^k V = V^{\otimes k} \Big/ \operatorname{span}\{ \quad \text{''} \quad + \quad \text{''} \quad \}$$

Remark $(\dim V = n < \infty)$

$$S^k V^\vee \cong \{\text{homogeneous poly of degree } k \text{ on } V\}$$

Note:

$$V^\vee = \{ f : V \to \Bbbk \text{ linear} \}$$

Let $\{e_1, \dots, e_n\}$ be a basis

$\rightsquigarrow \quad V \cong \Bbbk^n$

$$V^\vee \cong \operatorname{Hom}(\Bbbk^n, \Bbbk) = \{ f(x_1, \dots, x_n) = \underbrace{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n}_{\text{poly of deg } 1} \text{ for some } c_1, \dots, c_n \in \Bbbk \}$$

$$\phi : S^k V^\vee \longrightarrow \Bbbk[x_1, \dots, x_n]^k$$

$$= \{ \text{homogeneous poly of deg } k \text{ with coeff in } \Bbbk \}$$

n variables

$$f_1 \circ \cdots \circ f_k \mapsto f_1 \cdot f_2 \cdots f_k$$

Note: Φ is well-defined because

$$V^\vee \times \cdots \times V^\vee \longrightarrow \mathbb{k}[x_1, \dots, x_n]^k$$
$$(f_1, \dots, f_k) \mapsto f_1 \cdot f_2 \cdots f_k$$

is multilinear and symmetric

By checking bases or 1-1, onto,

Φ is an iso.

$$\Rightarrow \quad S^k V^\vee \cong \mathbb{k}[x_1, \dots, x_n]^k$$

#

§ Applications

§ Determinant

Theoretic definition of determinat:

Recall

$$\det : M_n(\mathbb{k}) \longrightarrow \mathbb{k}$$

$\leftrightsquigarrow$

$$D : \overbrace{\mathbb{k}^n \times \cdots \times \mathbb{k}^n}^{n} \longrightarrow \mathbb{k}$$

$$\left(\begin{pmatrix} a_{11} \\ \vdots \end{pmatrix} \quad \begin{pmatrix} a_{1n} \\ \vdots \end{pmatrix} \right) \mapsto \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots \end{pmatrix}$$

$$\left(\begin{pmatrix} \vdots \\ a_{n1} \end{pmatrix} \cdots \begin{pmatrix} \vdots \\ a_{nn} \end{pmatrix}\right) \mapsto \det\begin{pmatrix} \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

The map D is skew-symmetric, multilinear map

$$\det\left(\square \begin{matrix} ba_{1i}+c\tilde{a}_{1i} \\ \vdots \\ ba_{ni}+c\tilde{a}_{ni} \end{matrix} \square \right)$$

$$= b \cdot \det\left(\square \begin{matrix} a_{1i} \\ \vdots \\ a_{ni} \end{matrix} \square \right) + c \det\left(\square \begin{matrix} \tilde{a}_{1i} \\ \vdots \\ \tilde{a}_{ni} \end{matrix} \square \right)$$

$\Rightarrow \exists!$ linear map

$$\tilde{D} : \Lambda^n \mathbb{k}^n \longrightarrow \mathbb{k}$$

s.t. $\tilde{D}(\vec{v}_1 \wedge \cdots \wedge \vec{v}_n) = \det\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$

Recall: $\dim \Lambda^k \mathbb{k}^n = \binom{n}{k}$

<u>Note</u>:

$$\dim \Lambda^n \mathbb{k}^n = \binom{n}{n} = 1$$

So a linear

$$\Lambda^n \mathbb{k}^n \longrightarrow \mathbb{k}$$

is uniquely determined by its value

of

at

$$e_1 \wedge e_2 \wedge \cdots \wedge e_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

<u>Def</u>

$\det : M_n(\Bbbk^n) \to \Bbbk$ is the function

$$\tilde{D} \circ \varphi$$

where

$\varphi : M_n(\Bbbk^n) \to \Lambda^n \Bbbk^n$, $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}$

$\tilde{D}$ is the unique linear map

$\Lambda^n \Bbbk^n \to \Bbbk$ s.t.

$$\tilde{D}(e_1 \wedge \cdots \wedge e_n) = 1$$

<u>Prop</u>

The $\det : M_n(\Bbbk) \to \Bbbk$ is computed by the formula

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

<u>pf</u>

It suffices to check the map

$$\overbrace{\Bbbk^n \times \cdots \times \Bbbk^n}^{n} \to \Bbbk$$

$$\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right) \mapsto \boxed{\sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}}$$

satisfies (i) multilinear (ii) skew-symmetric

$$\text{(iii)} \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right) \mapsto 1$$

which are straightforward. #

§ Orientation "方向"

(V: vector space over $\mathbb{R}$, $\dim V = n < \infty$)

Note that

$$\Lambda^n V \cong \mathbb{R}$$

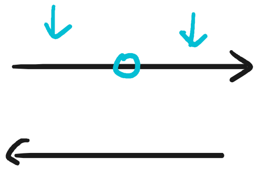
$\Rightarrow$ $\Lambda^n V \setminus \{0\}$ has 2 connected components

Def

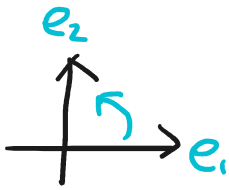
A choice of a connected component of $\Lambda^n V \setminus \{0\}$ is called an orientation of V

Example

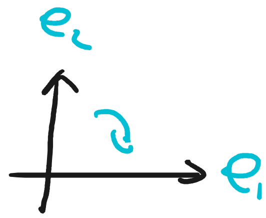
① $V = \mathbb{R}$ or



② $V = \mathbb{R}^2$



or



$e_1 \wedge e_2$ $\quad$ $e_2 \wedge e_1$

$$\Lambda^2 V = \Lambda^2 \mathbb{R}^2$$

$$e_1 \wedge e_2 = -e_2 \wedge e_1$$

Another description:

Let $\mathcal{F}(V) = \{\text{ordered bases for } V\}$

ordered

For

$$\beta = \{e_1, \cdots, e_n\} \in \mathcal{F}(V)$$

$$\gamma = \{\varepsilon_1, \cdots, \varepsilon_n\} \in \mathcal{F}(V).$$

$\exists$ $a_{ij} \in \mathbb{R}$ s.t.

$$e_i = \sum_{j=1}^{n} a_{ij}\,\varepsilon_j. \qquad \forall i = 1, \cdots, n$$

$\Rightarrow$ $\det(a_{ij}) \neq 0 \Rightarrow$ either $\det(a_{ij}) > 0$ or $\det(a_{ij}) < 0$

Define

$$\beta \sim \gamma \iff \det(a_{ij}) > 0.$$

__Lemma__

$\sim$ is an equivalence relation on $\mathcal{F}(V)$

An orientation is an equivalence class of $\mathcal{F}(V)/\sim$.

__Prop__

$$e_1 \wedge \cdots \wedge e_n = \det(a_{ij}) \cdot \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$$

in $\Lambda^n V$.

In particular, the 2 definitions of orientation are equivalent.

__pf__

$$e_1 \wedge \cdots \wedge e_n = \left(\sum_{j=1}^{n} a_{ij} \varepsilon_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} a_{ij} \varepsilon_j \right)$$

$$= \sum_{j_1, \cdots, j_n = 1}^{n} a_{ij_1} \cdots a_{ij_n} \cdot \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_n}$$

$$= \underbrace{\sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{1\sigma(n)} \cdot \underbrace{\boxed{\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(n)}}}_{= (-1)^{\sigma} \varepsilon_1 \wedge \cdots \wedge \varepsilon_n}}_{= \det(a_{ij})}$$

$$= \det(a_{ij}) \cdot \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \qquad \#$$

HW6

2. $A = \Bbbk[x_1, \cdots, x_k]$,

$\mu : A\otimes A \longrightarrow A$, $\mu(f\otimes g) = f g$

$T: V \to W$

$T': V' \to W'$

$\Rightarrow T\otimes T' : V\otimes V' \to W\otimes W'$

$(T\otimes T')(\vec{v}\otimes\vec{w}) = T(\vec{v})\otimes T'(\vec{w})$

(c) $\mu\circ(\mu\otimes \mathrm{id}_A) = \mu\circ(\mathrm{id}_A\otimes\mu)$

$$\mu\otimes\mathrm{id}_A : \underbrace{(A\otimes A)}_{\xrightarrow{\mu}\ A} \otimes \underbrace{A}_{\xrightarrow{\mathrm{id}_A}\ A} \longrightarrow A\otimes A$$

① $(\mu\otimes\mathrm{id}_A)((a\otimes b)\otimes c) = \mu(a\otimes b)\otimes \mathrm{id}_A(c)$

$= (a\cdot b)\otimes c \in A\otimes A$

② $(\mu\circ(\mu\otimes\mathrm{id}_A))((a\otimes b)\otimes c) = \mu((a\cdot b)\otimes c)$

$= \underbrace{(a\cdot b)\cdot c}\ \in A$

by associativity

③
$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes\mu"'] & A\otimes A \arrow[r, "\mu"] & A \\
& A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

$(\mu\circ(\mathrm{id}\otimes\mu))(a\otimes b\otimes c) = \mu(a\otimes(b\cdot c))$

$= \underbrace{a\cdot(b\cdot c)}$

So

$\mu\circ(\mathrm{id}_A\otimes\mu) = \mu\circ(\mu\otimes\mathrm{id}_A)$

$$\mu\circ(\mathrm{id}_A\otimes\mu) = \mu\circ(\mu\otimes \mathrm{id}_A)$$

($\Leftrightarrow$ μ is associative) #

(d) $\mu^i : A^{\otimes i+1} \longrightarrow A$, $\mu^i = \mu\circ(\mu^{i-1}\otimes \mathrm{id})$

$\mu^1 = \mu$

$\mu : A\otimes A\to A$, $\mu(a\otimes b) = a\cdot b$

$\mu^2 = \mu\circ(\mu\otimes \mathrm{id})$, $a\otimes b\otimes c \mapsto (a\cdot b)\cdot c = a\cdot b c$

$\mu^3 = \mu\circ(\mu^2\otimes \mathrm{id})$, $a\otimes b\otimes c\otimes d \mapsto (abc)\cdot d = abcd$

$\vdots$

$\mu^i(a_0\otimes\cdots\otimes a_i) = a_0\cdot a_1\cdots a_i$

Show

$$\mu\circ(\mu^i\otimes\mu^j) = \mu^{i+j+1} : A^{\otimes i+j+2}\to A$$

$$(\mu\circ(\mu^i\otimes\mu^j))(a_0\otimes\cdots\otimes a_i\otimes b_0\otimes\cdots\otimes b_j)$$

$$= \mu\left(\mu^i(a_0\otimes\cdots a_i)\otimes\mu^j(b_0\otimes\cdots b_j)\right)$$

$$= \mu\left((a_0\cdots a_i)\otimes(b_0\cdots b_j)\right)$$

$$= (a_0\cdots a_i)\cdot(b_0\cdots b_j)$$

...

$= \mu^{i+j+1}(a_0 \otimes \cdots \otimes a_i \otimes b_0 \otimes \cdots \otimes b_j)$ #

3. Let $\{\xi_1, \cdots, \xi_n\}$ be a linear indep set in $V^\vee$.
Show $\exists\ \vec{v}_1, \cdots, \vec{v}_n \in V$ s.t.

$$\xi_i(\vec{v}_j) = \begin{cases} 1 & , i = j \\ 0 & i \neq j. \end{cases}$$

<u>pf</u>

① Let

$$\phi : V \longrightarrow \mathbb{k}^n$$

$$\phi(\vec{v}) = (\xi_1(\vec{v}), \cdots, \xi_n(\vec{v}))$$

which is a linear map.

② Claim ϕ is onto,

If not, $\exists\ (a_1, \cdots, a_n) \in \mathbb{k}^n \setminus \operatorname{im}\phi.$

Subspace

linear

$\Rightarrow \exists\ \psi : \mathbb{k}^n \to \mathbb{k}$ s.t.

$$\psi(a_1, \cdots a_n) = 1$$

$$\psi(\operatorname{im}\phi) = 0$$

Take a basis for $\operatorname{im}\phi$ and extend

to a basis for $\mathbb{k}$
(add $(a_1, \cdots, a_n)$)

think about matrix representation of ψ w.r.t. standard bases

$c_1, \cdots, c_n$ are NOT all zeros

$$\Rightarrow \quad \psi(x_1, \cdots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

for some $c_1, \cdots, c_n \in \mathbb{k}$

$$V \xrightarrow{\Phi} \mathbb{k}^n \xrightarrow{\psi} \mathbb{k}$$

$$\Rightarrow 0 = \underline{\psi \circ \phi} = c_1 \xi_1 + c_2 \xi_2 + \cdots + c_n \xi_n$$

(~~$\to$~~ to linear indep.) $\cancel{\#}$

4.

$$\Phi : V^\vee \otimes W \longrightarrow \mathrm{Hom}(V, W).$$

$$\Phi(\xi \otimes \vec{w}) = \langle - \mid \xi \rangle \cdot \vec{w}$$

$$\Phi(\xi \otimes \vec{w})(\vec{v}) = \xi(\vec{v}) \cdot \vec{w}$$

Φ is linear and 1-1.

(a) Suppose $\dim V < \infty$. Show Φ is an iso.

pf

Let $f \in \mathrm{Hom}(V, W)$

Let $\{e_1, \cdots, e_n\}$ be a basis for V.

f is determined by

$$\vec{w}_i := f(e_i) \quad , \quad i = 1, \cdots, n.$$

$$\Rightarrow \quad f = \Phi\left(\sum_{j=1}^{n} \hat{e}_j \otimes \vec{w}_j \right)$$

where $\{\hat{e}_j\}$ is the dual basis,

because

$$\underbrace{\Phi\left(\sum_{j=1}^{n} \hat{e}_j \otimes \vec{w}_j\right)(e_i)}_{} = \sum_{j=1}^{n} \underbrace{\hat{e}_j(e_i)}_{= \begin{cases} 1 & \text{if } i=j \\ 0 & i \neq j \end{cases}} \cdot \vec{w}_j$$

$$= \vec{w}_i = \underline{\underline{f(e_i)}}$$

linear, coincide on a basis

$\Rightarrow$ they're same

$$\Rightarrow \Phi\left(\sum_{j=1}^{n} \hat{e}_j \otimes \vec{w}_j\right) = f$$

$\Rightarrow \Phi$ is onto #

(b) $V = W = \bigoplus_{\mathbb{N}} \mathbb{R}$

$\Rightarrow \mathrm{id}_V \in \mathrm{Hom}(V, V) \setminus \mathrm{im}(\Phi)$

because $\forall x \in V^\vee \otimes W$, $\exists\ \xi_i \in V^\vee$, $\vec{w}_i \in W$

s.t.

$$x = \sum_{i=1}^{n} \xi_i \otimes \vec{w}_i$$

$$\Rightarrow \Phi(x) = \sum_{i=1}^{n} \boxed{\Phi(\xi_i \otimes \vec{w}_i)}$$

$: \vec{v} \mapsto \xi_i(\vec{v}) \cdot \vec{w}_i$

$\in \mathrm{Span}\{\vec{w}_i\}$

$$\Phi(x)(\vec{v}) = \sum_{i=1}^{n} \xi_i(\vec{v}) \cdot \vec{w}_i \in \mathrm{Span}\{\vec{w}_1, \dots, \vec{w}_n\}$$

$\forall \vec{v} \in V$

$\forall\, v \in V$

$$\operatorname{im}(\Phi(x)) \subseteq \underbrace{\operatorname{Span}\{\vec{w}_1, \cdots, \vec{w}_n\}}_{\dim \le n}$$

But

$$\operatorname{im}(\mathrm{id}_V) = \underbrace{V = \bigoplus_{\mathbb{N}} \mathbb{R}}_{\dim = \infty \not\le n}$$

So

$$\Phi(x) \neq \mathrm{id}_V \qquad \forall\, x \in V^\vee \otimes W$$

#