Linear Algebra 11/2

§ Symmetric tensor product

Recall

In $V \otimes V$,

$\vec{v}_1 \otimes \vec{v}_2 \neq \vec{v}_2 \otimes \vec{v}_1$ in general

If you want a commutative product, you can do …

Def.

The symmetric tensor product $V \odot V$ is the quotient space

$$S^2V = V \odot V = V \otimes V \Big/ \operatorname{span}\{x \otimes y - y \otimes x \mid x, y \in V\}$$

which is also denoted by S^2V (or $\mathrm{Sym}^2 V$)

More generally,

$$S^kV = \overbrace{V \otimes \cdots \otimes V}^{k \text{ times}} \Big/ \operatorname{span}\left\{ \vec{v}_1 \otimes \cdots \otimes \vec{v}_{i-1} \otimes (\vec{v}_i \otimes \vec{v}_{i+1} - \vec{v}_{i+1} \otimes \vec{v}_i) \otimes \cdots \otimes \vec{v}_k \mid \right\}$$

We denote $\qquad \vec{v}_1, \dots, \vec{v}_k \in V$

$$\vec{v}_1 \odot \cdots \odot \vec{v}_k = [\vec{v}_1 \otimes \cdots \otimes \vec{v}_k] \in S^k V$$

<u>Remark</u>

In $S^k V$, we have

$$\vec{v}_1 \odot \cdots \odot \vec{v}_i \odot \vec{v}_{i+1} \odot \cdots \odot \vec{v}_k$$

$$= \vec{v}_1 \odot \cdots \odot \vec{v}_{i+1} \odot \vec{v}_i \odot \cdots \odot \vec{v}_k$$

$$= \vec{v}_{\sigma(1)} \odot \cdots \odot \vec{v}_{\sigma(k)}$$

$\forall\ \sigma \in S_k =$ permutation group.

<u>Recall</u>:

- $S_k = \{ \sigma : \{1, \dots, k\} \to \{1, \dots, k\}$ 1-1, onto maps $\}$
- $\sigma = (i_1\ \ i_1+1)(i_2\ \ i_2+1) \cdots (i_N\ \ \underset{\neq k}{i_N+1})$

  for some $i_1, \dots, i_N \in \{1, \dots, k-1\}$

  NOT fixed by σ but (even/odd) parity of N is fixed by σ
- σ is <u>even</u> if N is even
  <u>odd</u> if N is odd

- $(-1)^\sigma := \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$

<u>Example</u>

Let e_1, e_2 be the standard basis for $\mathbb{R}^2$

In $S^2\mathbb{R}^2$,

$$(e_1+2e_2)\odot(3e_1+4e_2)$$

$$e_1\odot e_2 = \boxed{e_2\odot e_1}$$

$$= 3e_1\odot e_1 + 4e_1\odot e_2 + 6\,\boxed{e_2\odot e_1} + 8e_2\odot e_2$$

$$= 3e_1\odot e_1 + 10\,e_1\odot e_2 + 8\,e_2\odot e_2$$

∎

Let $\varphi: \overbrace{V\times\cdots\times V}^{k} \longrightarrow S^kV$ be the map

$$\varphi(\vec v_1,\dots,\vec v_k) = \vec v_1\odot\cdots\odot\vec v_k$$

which is k-linear and symmetric, i.e.,

$$\varphi(\vec v_{\sigma(1)},\dots,\vec v_{\sigma(k)}) = \varphi(\vec v_1,\dots,\vec v_k) \quad \forall\sigma\in S_k.$$

Prop

If $f: \overbrace{V\times\cdots\times V}^{k} \longrightarrow W$ is a multilinear map with the property

$$f(\vec v_{\sigma(1)},\dots,\vec v_{\sigma(k)}) = f(\vec v_1,\dots,\vec v_k) \quad \forall \vec v_1,\dots,\vec v_k\in V,\ \forall\sigma\in S_k,$$

then $\exists!$ linear map

$$\tilde f: S^kV \longrightarrow W$$

s.t. $\overbrace{V\times\cdots\times V}^{k} \xrightarrow{\forall f} W$

$$\begin{tikzcd} V\times\cdots\times V \arrow[r] \arrow[d, "\varphi"'] & W \\ S^kV \arrow[ur, dashed, "\exists!\,\tilde f"'] & \end{tikzcd}$$

commutes.

Remark

So we have

$$\left\{ \overbrace{V\times\cdots\times V}^{k} \xrightarrow{\tilde f\circ\varphi} W \ \text{multi linear symmetric} \right\} \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} \left\{ S^kV \xrightarrow{\tilde f} W \ \text{linear} \right\}$$

pf of Prop

$$\begin{tikzcd} V\times\cdots\times V \arrow[d] \arrow[r, "\forall f", "\text{k-linear, symmetric}"'] & W \\ V^{\otimes k} \arrow[d] \arrow[ur, dashed, "\exists!\ \text{linear}\ \hat f"'] & \\ V^{\otimes k}/R = S^kV \arrow[uur, dashed, "\exists!\ \tilde f\ \text{linear}"'] & \end{tikzcd}$$

$\hat f(R)=0$

#

$$\overbrace{V \times \cdots \times V}^{k} \longrightarrow V^{\otimes k}$$

$$(\vec{v}_1, \cdots, \vec{v}_k) \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \vec{v}_{\sigma(1)} \otimes \vec{v}_{\sigma(2)} \otimes \cdots \otimes \vec{v}_{\sigma(k)}$$

is multilinear and symmetric

e.g. $(\vec{v}_1, \vec{v}_2) \mapsto \frac{1}{2}(\vec{v}_1 \otimes \vec{v}_2 + \vec{v}_2 \otimes \vec{v}_1)$

$(\vec{v}_2, \vec{v}_1) \mapsto \frac{1}{2}(\vec{v}_2 \otimes \vec{v}_1 + \vec{v}_1 \otimes \vec{v}_2)$

So $\exists!$ linear map $\mathrm{sym}: S^k V \to V^{\otimes k}$ s.t.

$$\mathrm{sym}(\vec{v}_1 \odot \cdots \odot \vec{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \vec{v}_{\sigma(1)} \otimes \cdots \otimes \vec{v}_{\sigma(k)}$$

Note

- $\mathrm{sym}: S^k V \longrightarrow V^{\otimes k}$ is well-defined
- $\pi \circ \mathrm{sym} = \mathrm{id}_{S^k V}$, where

  $\pi: V^{\otimes k} \longrightarrow S^k V = V^{\otimes k}/R$, $\pi(x) = [x]$

  is the quotient map

  because

  $$(\pi \circ \mathrm{sym})(\vec{v}_1 \odot \cdots \odot \vec{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \underbrace{\vec{v}_{\sigma(1)} \odot \cdots \odot \vec{v}_{\sigma(k)}}_{= \vec{v}_1 \odot \cdots \odot \vec{v}_k}$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \vec{v}_1 \odot \cdots \odot \vec{v}_k$$

$$= \frac{1}{k!} \cdot k! \cdot \vec{v}_1 \odot \cdots \odot \vec{v}_k = \mathrm{id}(\vec{v}_1 \odot \cdots \odot \vec{v}_k)$$

$\therefore\ \pi \circ \underset{1\text{-}1}{\underline{\mathrm{sym}}} = \mathrm{id} \quad \Leftarrow 1\text{-}1$

Prop

The map $\mathrm{sym}: S^k V \longrightarrow V^{\otimes k}$ is one-to-one

An alternative definition of $S^k V$

Let $\sigma \in S_k$. Since the map

$$\overbrace{V \times \cdots \times V}^{k} \longrightarrow V^{\otimes k}$$

$$(\vec{v}_1, \cdots, \vec{v}_k) \longmapsto \vec{v}_{\sigma(1)} \otimes \vec{v}_{\sigma(2)} \otimes \cdots \otimes \vec{v}_{\sigma(k)}$$

is multilinear, $\exists!$ linear map

$$T_\sigma : V^{\otimes k} \longrightarrow V^{\otimes k}$$

s.t.

$$T_\sigma(\vec{v}_1 \otimes \cdots \otimes \vec{v}_k) = \vec{v}_{\sigma(1)} \otimes \cdots \otimes \vec{v}_{\sigma(k)}$$

Let $\{e_i \mid i \in I\}$ be a basis for V.

An element $A \in V^{\otimes k}$ is of the form

$$A = \sum_{\substack{(i_1,\dots,i_k)\in I^k \\ \text{finite sum}}} a_{i_1,\dots,i_k}\, e_{i_1}\otimes\cdots\otimes e_{i_k}$$

$\Rightarrow$

$$T_\sigma(A) = \sum a_{i_1,\dots,i_k}\, e_{i_{\sigma(1)}}\otimes\cdots\otimes e_{i_{\sigma(k)}}$$

$$= \sum a_{i_{\sigma^{-1}(1)},\dots,i_{\sigma^{-1}(k)}} \cdot e_{i_1}\otimes\cdots\otimes e_{i_k}$$

Prop

$$\mathrm{Sym}(S^kV) = \underbrace{\{A\in V^{\otimes k} \mid T_\sigma(A) = A \ \forall \sigma\in S_k\}}_{\overset{\shortparallel}{S}}$$

$$= \{A\in V^{\otimes k} \mid a_{i_1,\dots,i_k} = a_{i_{\sigma(1)},\dots,i_{\sigma(k)}} \ \forall\sigma\in S_k\}$$

pf

$\mathrm{Sym}(S^kV) \subseteq S$: direct computation

check: $\forall \vec v_1,\dots,\vec v_k\in V$.

$$T_\sigma(\mathrm{sym}(\vec v_1\odot\cdots\odot\vec v_k)) = \mathrm{sym}(\vec v_1\odot\cdots\odot\vec v_k) \checkmark$$

$$\shortparallel$$

$$T_\sigma\Big(\frac{1}{k!}\sum_{\alpha\in S_k}\vec v_{\alpha(1)}\otimes\cdots\otimes\vec v_{\alpha(k)}\Big)$$

$\{\alpha\circ\sigma \mid \alpha\in S_k\}$

$= \{\alpha \mid \alpha\in S_k\}$

$$= \frac{1}{k!}\sum_{\alpha\in S_k}\vec v_{\alpha(\sigma(1))}\otimes\cdots\otimes\vec v_{\alpha(\sigma(k))}$$

$$\frac{1}{k!}\sum \vec v\otimes\cdots\otimes\vec v = \mathrm{Sym}(\vec v_1\otimes\cdots\otimes\vec v_k)$$

$$= \frac{1}{k!} \sum_{\alpha \in S_k} v_{\alpha(1)} \otimes \cdots \otimes v_{\alpha(k)} = \mathrm{sym}(v_1 \otimes \cdots \otimes v_k)$$

<u>$\mathcal{S} \subseteq \mathrm{sym}(S^k V)$</u> :

$$\forall A = \sum a_{i_1,\dots,i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \in \mathcal{S}$$

one can check:

$$i_1 = \cdots = i_{m_1} < i_{m_1+1} = \cdots = i_{m_1+m_2} < \cdots\ i_k$$

$$A = \mathrm{sym}\Big(\sum_{i_1 \le \cdots \le i_k} \frac{k!}{m_1! \cdots m_p!}\, a_{i_1,\dots,i_k}\, e_{i_1} \odot \cdots \odot e_{i_k} \Big)$$

$$\in \mathrm{sym}(S^k V)$$

#

e.g. $A = e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1$

$T_\sigma(A) = A$ ✓

$\rightsquigarrow$

$$\mathrm{sym}(e_1 \otimes e_2 \otimes e_2)$$

$$= \frac{1}{6!} \Big(\overset{1}{e_1} \otimes \overset{2}{e_2} \otimes \overset{3}{e_2} + \overset{2}{e_2} \otimes \overset{1}{e_1} \otimes \overset{3}{e_2} + \overset{2}{e_2} \otimes \overset{3}{e_2} \otimes \overset{1}{e_1}$$
$$+\ \overset{1}{e_1} \otimes \overset{3}{e_2} \otimes \overset{2}{e_2} + \overset{3}{e_2} \otimes \overset{1}{e_1} \otimes \overset{2}{e_2} + \overset{3}{e_2} \otimes \overset{2}{e_2} \otimes \overset{1}{e_1} \Big)$$

$$= \frac{1}{3} \left(e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 \right)$$

$$= \frac{1}{3} A$$

Some people $\mathcal{S} = \mathrm{sym}(S^k V)$ as the definition of S^kV, and an element in $\mathrm{sym}(S^kV)$ is called a symmetric tensor

By Prop,

$$\mathrm{sym}: S^kV \longrightarrow \underset{\substack{\| \\ \mathrm{Sym}^k V}}{\mathrm{sym}(S^kV)} \subseteq V^{\otimes k}$$

is an isomorphism.

Prop

Let $\{e_1, \cdots, e_n\}$ be a basis for V.

Then

$$\{ e_{i_1} \odot \cdots \odot e_{i_k} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n \}$$

is a basis ~~for~~ S^kV.

$$\overbrace{\circ \cdots \circ}^{n} \quad \overbrace{| \cdots |}^{k-1}$$

$$n = 1 + \cdots + 1$$

In particular,

$$\dim S^kV = \binom{n+k-1}{k}$$

§ Skew-symmetric tensor product

Def

$$\vec{v}_1 \odot \vec{v}_2 = \vec{v}_2 \odot \vec{v}_1$$

$$\vec{v}_1 \wedge \vec{v}_2 = -\vec{v}_2 \wedge \vec{v}_1$$

$$V \wedge V := V \otimes V \Big/ \operatorname{span}\{x \otimes y + y \otimes x \mid x, y \in V\}$$

which is also denoted by $\Lambda^2 V$

skew-symmetric tensor product

or wedge product

More generally,

$$\Lambda^k V := V^{\otimes k} \Big/ \operatorname{span}\{\vec{v}_1 \otimes \cdots (\vec{v}_i \otimes \vec{v}_{i+1} + \vec{v}_{i+1} \otimes \vec{v}_i) \otimes \cdots \vec{v}_k\}$$

$$\vec{v}_1 \wedge \cdots \wedge \vec{v}_k := [\vec{v}_1 \otimes \cdots \otimes \vec{v}_k] \in \Lambda^k V = V^{\otimes k} / R$$

Remark

In $\Lambda^k V$,

$$\vec{v}_1 \wedge \cdots \underline{\vec{v}_i \wedge \vec{v}_{i+1}} \wedge \cdots \vec{v}_k = - \vec{v}_1 \wedge \cdots \underline{\vec{v}_{i+1} \wedge \vec{v}_i} \cdots \wedge \vec{v}_k$$

$$= (-1)^{\sigma} \cdot \vec{v}_{\sigma(1)} \wedge \cdots \wedge \vec{v}_{\sigma(k)} \qquad \forall \sigma \in S_k$$

Also note that

$$\vec{v} \wedge \vec{v} = - \vec{v} \wedge \vec{v}$$

$$\Rightarrow \quad 2 \cdot \vec{v} \wedge \vec{v} = 0 \quad \Rightarrow \quad \vec{v} \wedge \vec{v} = 0 \quad (\text{if } \operatorname{char} \mathbb{k} \neq 2)$$

Example

In $\Lambda^2\mathbb{R}^2$,

$$(e_1+2e_2)\wedge(3e_1+4e_2)$$

$$= 3\overset{0}{\boxed{e_1\wedge e_1}} + 4e_1\wedge e_2 + 6\overset{-e_1\wedge e_2}{\boxed{e_2\wedge e_1}} + 8\overset{0}{e_2\wedge e_2}$$

$$= -2\cdot e_1\wedge e_2$$

Prop

If $f: \overbrace{V\times\cdots\times V}^{k} \to W$ is multilinear and

$$f(\vec{v}_1,\dots,\vec{v}_k) = (-1)^{\sigma} f(\vec{v}_{\sigma(1)},\dots,\vec{v}_{\sigma(k)}) \quad \forall \sigma\in S_k$$

i.e. skew-symmetric

then $\exists!$ linear $\tilde{f}: \Lambda^k V \to W$ s.t.

$$\begin{tikzcd}
(\vec{v}_1,\dots,\vec{v}_k) \arrow[d, mapsto] & \overbrace{V\times\cdots\times V}^{k} \arrow[r, "\forall f"', "\substack{\text{multilinear}\\\text{skew-symm}}"] \arrow[d] & W \\
\vec{v}_1\wedge\cdots\wedge\vec{v}_k & \Lambda^k V \arrow[ur, dashed, "\substack{\exists!\,\tilde{f}\\\text{linear}}"'] &
\end{tikzcd}$$

Commutes

Example

$\exists!$ linear map

$$\widetilde{\mathrm{Sym}}: \Lambda^k V \longrightarrow V^{\otimes k}$$

s.t.

$$\widetilde{\mathrm{Sym}}(\vec{v}_1 \wedge \cdots \wedge \vec{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} \vec{v}_{\sigma(1)} \otimes \cdots \otimes \vec{v}_{\sigma(k)}$$

because $V \times \cdots \times V \to V^{\otimes k}$

$$(\vec{v}_1, \cdots, \vec{v}_k) \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} \vec{v}_{\sigma(1)} \otimes \cdots \otimes \vec{v}_{\sigma(k)}$$

is skew-symm multilinear

<u>Prop</u>

$\widetilde{\mathrm{Sym}} : \Lambda^k V \to V^{\otimes k}$ is one-to-one

<u>Prop</u>

another definition

$$\widetilde{\mathrm{Sym}}(\Lambda^k V) = \{ A \in V^{\otimes k} \mid T_\sigma(A) = (-1)^{\sigma} \cdot A \ \forall \sigma \in S_k \}$$

$\Downarrow$

A is called <u>skew-symmetric tensor</u>

<u>Prop</u> (Suppose char $\mathbb{k} \neq 2$)

Let $\{e_1, \cdots, e_n\}$ be a basis for V.

Then

$$\{ e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 \underset{\neq}{\leq} i_2 < \cdots < i_k \leq n \}$$

is a basis for $\Lambda^k V$ (Recall $e_1 \wedge e_2 \wedge e_2 = 0$)

In particular,

$$\dim \Lambda^k V = \binom{n}{k}.$$

§ Pairings

Let $f, g : V \to \Bbbk$.

$f \circ g : S^2 V \to \Bbbk \quad = \ ?$

$f \wedge g : \Lambda^2 V \to \Bbbk \quad = \ ?$

Problem:

The formula:

$$(f\circ g)(\vec{v} \circ \vec{w}) = f(\vec{v}) \cdot g(\vec{w})$$

is NOT well-defined !! $\neq$

$$(f\circ g)(\vec{w} \circ \vec{v}) = f(\vec{w}) \cdot g(\vec{v})$$

Recall we have

$$\langle \cdot | \cdot \rangle : V \otimes V^\vee \longrightarrow \Bbbk, \quad \langle \vec{v} | f \rangle = f(\vec{v})$$

or

$$\langle \cdot | \cdot \rangle : V^\vee \otimes V \longrightarrow \Bbbk, \quad \langle f | \vec{v} \rangle = f(\vec{v})$$

This induces

$$(V^\vee)^{\otimes k} \times V^{\otimes k} \longrightarrow \Bbbk$$

$$\langle f_1 \otimes \cdots \otimes f_k \,|\, \vec{v}_1 \otimes \cdots \otimes \vec{v}_k \rangle = \langle f_1 | \vec{v}_1 \rangle \cdot \langle f_2 | \vec{v}_2 \rangle \cdots \langle f_k | \vec{v}_k \rangle$$

$$= f_1(\vec{v}_1) \cdot \cdots f_k(\vec{v}_k) \quad \leftarrow \text{multilinear}$$

$\rightsquigarrow$

$$\langle \cdot | \cdot \rangle : S^k(V^\vee) \times S^k V \longrightarrow \Bbbk$$

$$\langle f_1 \odot \cdots \odot f_k \mid \vec{v}_1 \odot \cdots \odot \vec{v}_k \rangle$$

$k!\,\mathrm{sym}(\vec{v}_1 \odot \cdots \odot \vec{v}_k)$

$$= \langle f_1 \otimes \cdots \otimes f_k \mid \sum_{\sigma \in S_k} \vec{v}_{\sigma(1)} \otimes \cdots \otimes \vec{v}_{\sigma(k)} \rangle$$

$$= \sum_{\sigma \in S_k} f_1(\vec{v}_{\sigma(1)}) \cdot f_2(\vec{v}_{\sigma(2)}) \cdots f_k(\vec{v}_{\sigma(k)})$$

<u>Lemma</u>

This $\langle \cdot \mid \cdot \rangle$ is well-defined.

Similarly, one has

$$\langle \cdot \mid \cdot \rangle : \Lambda^k(V^\vee) \times \Lambda^k V \longrightarrow \mathbb{k}$$

with the property

<u>Lemma</u>

This is well-defined

$$\langle f_1 \wedge f_2 \wedge \cdots \wedge f_k \mid \vec{v}_1 \wedge \cdots \wedge \vec{v}_k \rangle$$

$$:= \langle f_1 \otimes \cdots \otimes f_k \mid \sum_{\sigma \in S_k} (-1)^\sigma \vec{v}_{\sigma(1)} \otimes \cdots \otimes \vec{v}_{\sigma(k)} \rangle$$

$$= \sum_{\sigma \in S_k} (-1)^\sigma f_1(\vec{v}_{\sigma(1)}) \cdots f_k(\vec{v}_{\sigma(k)})$$

<u>Prop</u>

Suppose $\dim V < \infty$. Then

$$S^k(V^\vee) \xrightarrow{\cong} (S^k V)^\vee, \quad \Theta \mapsto \langle \Theta \mid - \rangle$$

$$\Lambda^k(V^\vee) \xrightarrow{\cong} (\Lambda^k V)^\vee, \quad \omega \mapsto \langle \omega \mid - \rangle$$

are isomorphisms.

$$S^k V = \text{linear diff. op. on } V \text{ (constant coeff, order } k)$$

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}$$

Remark

$$S^k(V^\vee) \cong \left\{ \begin{array}{c} \text{homogeneous polynomials} \\ \text{of degree } k \text{ on } V \end{array} \right\}$$

e.g.

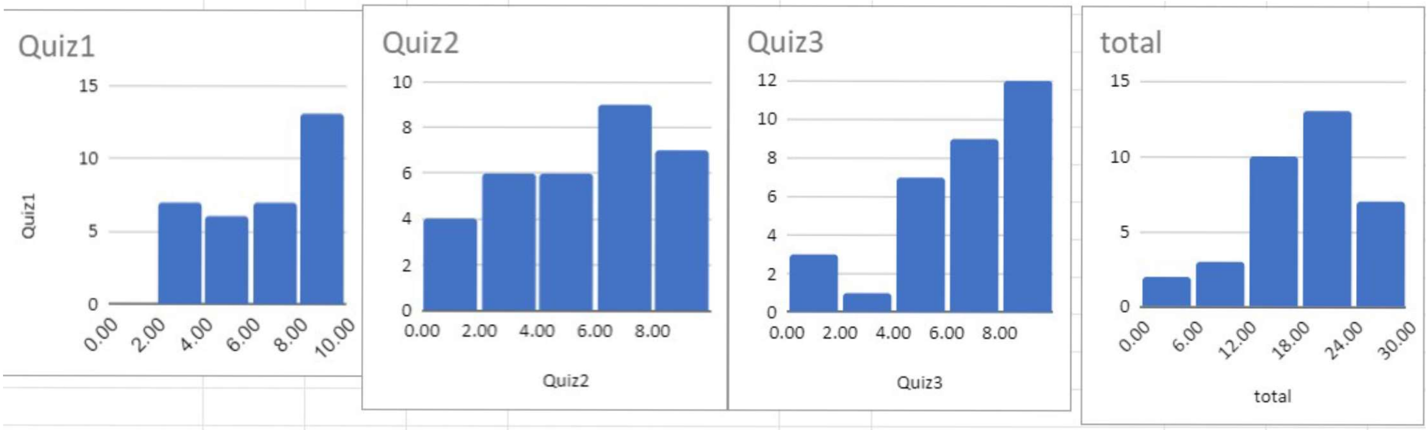
$f \in S^1(V^\vee) = V^\vee$, $\quad f : V \to \Bbbk$ is linear

With a basis, $\quad x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in V$

$a_i = f(e_i)$

$$f(x_1, \cdots, x_n) = a_1 x_1 + \cdots + a_n x_n$$

for some $a_1, \cdots, a_k \in \Bbbk$ $\quad [f] = (\underset{\underset{f(e_1)}{\parallel}}{a_1} \cdots a_n)$



Average	17.85714		6.342857	5.485714	6.205882
Average (except 0)	17.85714		6.342857	5.818181	6.806451
Quartile0	3		2	0	0
Quartile1	14		4	3	5
Quartile2	18		7	6	6
Quartile3	21.5		8	8	8
Quartile4	28		10	10	10
標準差	6.399185		2.436763	2.863270	2.637485
非零數	35		35	33	31
不到總分50%人數	35		11	12	5
滿分/A+人數	~~0~~		2	3	2

HW3. 4. b

Given $a^{j}_{i_1, i_2, i_3} \in k$, show that $\exists$ multilinear map $f: V_1 \times V_2 \times V_3 \to W$ whose coordinate representation is $(a^{j}_{i_1, i_2, i_3})$

Assume
$\vec{v}^{\,1}_1, \dots, \vec{v}^{\,1}_{n_1}$ basis for V_1
$\vec{v}^{\,2}_1, \dots, \vec{v}^{\,2}_{n_2}$ " V_2
$\vec{v}^{\,3}_1, \dots, \vec{v}^{\,3}_{n_3}$ " V_3
$\vec{w}_1, \dots, \vec{w}_m$ " W.

For $x \in V_1$, $y \in V_2$, $z \in V_3$, $\exists!\ \alpha_{i_1}, \beta_{i_2}, \gamma_{i_3} \in k$

s.t. $x = \sum_{i_1=1}^{n_1} \alpha_{i_1} \vec{v}^{\,1}_{i_1}$, $y = \sum_{i_2=1}^{n_2} \beta_{i_2} \vec{v}^{\,2}_{i_2}$, $z = \sum_{i_3=1}^{n_3} \gamma_{i_3} \vec{v}^{\,3}_{i_3}$

Define

$$f : V_1 \times V_2 \times V_3 \to W$$

by

$$f(x,y,z) = f\left(\sum \alpha_{i_1} \vec{v}^{\,1}_{i_1}, \sum \beta_{i_2} \vec{v}^{\,2}_{i_2}, \sum \gamma_{i_3} \vec{v}^{\,3}_{i_3}\right)$$

$$:= \sum_{i_1, i_2, i_3, j} \alpha_{i_1} \beta_{i_2} \gamma_{i_3} \boxed{a^{j}_{i_1 i_2 i_3}} \vec{w}_j$$

|| hope

coeff of $\vec{w}_j$ in $f(\vec{v}^{\,1}_{i_1}, \vec{v}^{\,2}_{i_2}, \vec{v}^{\,3}_{i_3})$

Then

<u>Check</u> f is multilinear