Linear Algebra 10/26

Summary of tensor product

① Any element $x \in V \otimes W$ is of the form

$$\boxed{x = \sum_{i=1}^{k} \vec{v}_i \otimes \vec{w}_i}$$

for some $\vec{v}_1, \cdots, \vec{v}_k \in V$, $\vec{w}_1, \cdots, \vec{w}_k \in W$.

② $(a\vec{v}_1 + b\vec{v}_2) \otimes \vec{w} = a\, \vec{v}_1 \otimes \vec{w} + b\, \vec{v}_2 \otimes \vec{w}$

$\vec{v} \otimes (a\, \vec{w}_1 + b\, \vec{w}_2) = a\, \vec{v} \otimes \vec{w}_1 + b\, \vec{v} \otimes \vec{w}_2$

③ Let β be a basis for V

γ be a basis for W

Then $\gamma \times \beta = \{\vec{e} \otimes \vec{\varepsilon} \mid \vec{e} \in \beta, \vec{\varepsilon} \in \gamma\}$ is a basis for $V \otimes W$

$$\boxed{V \otimes W \cong \mathbb{k}^{(\beta \times \gamma)}}$$

⚠ NOT good

one way to describe

for modules!!

$V \otimes W$ for vector spaces

For example, if

$\vec{e}_1, \dots, \vec{e}_n$: basis for V

$\vec{\varepsilon}_1, \dots, \vec{\varepsilon}_m$: basis for W

$\Rightarrow$

$$\begin{matrix} \vec{e}_1 \otimes \vec{\varepsilon}_1, & \vec{e}_1 \otimes \vec{\varepsilon}_2, & \dots, & \vec{e}_1 \otimes \vec{\varepsilon}_m, \\ \vec{e}_2 \otimes \vec{\varepsilon}_1, & \vec{e}_2 \otimes \vec{\varepsilon}_2, & \dots, & \vec{e}_2 \otimes \vec{\varepsilon}_m, \\ \dots & & \dots, & \\ \vec{e}_n \otimes \vec{\varepsilon}_m, & & \dots & \vec{e}_n \otimes \vec{\varepsilon}_m \end{matrix}$$

$\dim(V \otimes W) = \dim V \cdot \dim W$

form a basis for $V \otimes W$

④ Universal property:

$$\begin{tikzcd} (\vec{v}, \vec{w}) \arrow[d, mapsto] & V \times W \arrow[r, "\forall \text{ bilinear } f"] \arrow[d] & W' \\ \vec{v} \otimes \vec{w} & V \otimes W \arrow[ur, dashed, "\exists! \text{ linear } \tilde{f}"'] & \end{tikzcd}$$

Last week: use it to show

$V \otimes W \cong \underline{\underline{\Bbbk^{(\beta \times \gamma)}}}$ ← satisfies the same universal property

property

Today (simpler, important): use it to obtain linear maps $V \otimes W \longrightarrow W'$

Problem: $V \otimes W$ is a quotient space.

A formula may have a well-def problem!!

e.g. ① HW4.2, ② HW5.4 — complexification

③ $f: \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$, $f(a \otimes b) = a+b$ is NOT well-defined

because $(a,b) \mapsto a+b$ is NOT bilinear

$1 = f(1 \otimes 0) = f(1 \otimes 2 \cdot 0) = f(2 \cdot 1 \otimes 0)$

$= f(2 \otimes 0) = 2 \quad \rightarrow\leftarrow$

§ Tensor product and linear map

Let $f: V_1 \to W_1$, $g: V_2 \to W_2$ be linear.

Since the map

$$\begin{aligned} V_1 \times V_2 &\longrightarrow W_1 \otimes W_2 \\ (\vec{v}_1, \vec{v}_2) &\longmapsto f(\vec{v}_1) \otimes g(\vec{v}_2) \end{aligned}$$

is bilinear map, by Universal Property,

$\exists !$ linear

$$f \otimes g: V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$

s.t.

$$(f\otimes g)(\vec{v}_1\otimes\vec{v}_2) = f(\vec{v}_1)\otimes g(\vec{v}_2)$$

" tensor product of linear maps "

Prop (Adjoint functor property)

$$\mathrm{Hom}(V_1\otimes V_2, W) \cong \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W))$$

pf

universal property

$$\mathrm{Hom}(V_1\otimes V_2, W)$$

$$\cong \left\{ \text{bilinear maps } V_1\times V_2 \longrightarrow W \right\}$$

f G

$(\vec{v}_1, \vec{v}_2) \mapsto f_{\vec{v}_1}(\vec{v}_2)$

$$\cong \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W))$$

$\vec{v}_1 \mapsto f(\vec{v}_1, -)$ $\qquad \Phi: \vec{v}_1 \mapsto f_{\vec{v}_1}$

$\blacktriangle$

Prop

Let

$$\Phi : V^\vee\otimes W \longrightarrow \mathrm{Hom}(V, W)$$

be the linear map s.t.

$$\Phi(\xi\otimes\vec{w}) = (\vec{v}\mapsto \xi(\vec{v})\cdot\vec{w})$$

$V^\vee\times W \longrightarrow \mathrm{Hom}(V,W)$

$(\xi, \vec{w}) \mapsto \xi(-)\cdot\vec{w}$

is bilinear (universal property)

The linear map Φ is 1-1.
Furthermore, if one of V and W is finite-dimensional, then Φ is an isomorphism:

$$V^{\vee} \otimes W \cong \mathrm{Hom}(V, W)$$

pf: skip.

§ Product properties

cf. $(\mathbb{N}, +, \cdot)$

(vector space, $\oplus$, $\otimes$) $\nearrow$ dim

Prop

① $V \otimes W \cong W \otimes V$

② $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \cong V_1 \otimes V_2 \otimes V_3$

③ $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$

Remark

In general, for $\vec{v}_1, \vec{v}_2 \in V$,

$$\vec{v}_1 \otimes \vec{v}_2 \neq \vec{v}_2 \otimes \vec{v}_1$$

eg. In $\mathbb{R}^2 \otimes \mathbb{R}^2$, $\{\vec{e}_1 \otimes \vec{e}_2, \vec{e}_2 \otimes \vec{e}_1\}$ is linearly

($\vec{e}_1, \vec{e}_2$: standard basis for $\mathbb{R}^2$) independent

In particular, $\vec{e}_1 \otimes \vec{e}_2 \neq \vec{e}_2 \otimes \vec{e}_1$

Sketch of pf

Step1 construct iso (candidates) By universal property.

① $V \otimes W \xrightarrow{\exists! \text{ linear map s.t.}} W \otimes V : \vec{v} \otimes \vec{w} \mapsto \vec{w} \otimes \vec{v}$

② $V_1 \otimes V_2 \otimes V_3 \xrightarrow{\exists!} (V_1 \otimes V_2) \otimes V_3 : \vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3 \mapsto (\vec{v}_1 \otimes \vec{v}_2) \otimes \vec{v}_3$

$V_1 \otimes V_2 \otimes V_3 \xrightarrow{\exists!} V_1 \otimes (V_2 \otimes V_3) : \vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3 \mapsto \vec{v}_1 \otimes (\vec{v}_2 \otimes \vec{v}_3)$

③ $(V_1 \oplus V_2) \otimes W \xrightarrow{\exists!} (V_1 \otimes W) \oplus (V_2 \otimes W)$:

$(\vec{v}_1, \vec{v}_2) \otimes \vec{w} \longmapsto (\vec{v}_1 \otimes \vec{w}, \vec{v}_2 \otimes \vec{w})$

Step2: Compare basis or construct an inverse

#

2. (4 points) Let β_α be a basis for V_α, $\alpha \in I$. Show that the disjoint union $\beta = \coprod_{\alpha\in I} \beta_\alpha$ is a basis for the direct sum $\bigoplus_{\alpha\in I} V_\alpha$.

① β is linearly independent:

Suppose $x_1, \cdots, x_k \in \beta$, $a_1, \cdots, a_k \in k$

$$(*) \quad a_1x_1 + \cdots + a_kx_k = 0 \quad \leftarrow \text{in } \bigoplus_{\alpha\in I} V_\alpha$$

Since $\beta = \coprod_{\alpha\in I} \beta_\alpha$, $x_i \in \beta_\alpha$ for some α.

Let $x_1^\alpha, \cdots, x_{i_\alpha}^\alpha \in \beta_\alpha \cap \{x_1, \cdots, x_k\}$

Then $(*) \iff$

finite sum in $\bigoplus_{\alpha\in I} V_\alpha$ $\rightarrow$ $$\sum_\alpha \underbrace{(a_1^\alpha x_1^\alpha + \cdots + a_{i_\alpha}^\alpha x_{i_\alpha}^\alpha)}_{\in V_\alpha} = 0$$

$$\Rightarrow \quad a_1^\alpha x_1^\alpha + \cdots + a_{i_\alpha}^\alpha x_{i_\alpha}^\alpha = \underset{\text{in } V_\alpha}{0} \quad \forall \alpha \in I$$

Since $x_1^\alpha, \cdots, x_{i_\alpha}^\alpha \in \beta_\alpha$ and β_α is linearly indep.

we have

$$a_1^\alpha = \cdots = a_{i_\alpha}^\alpha \quad \forall \alpha$$

$$\Rightarrow \quad a_1 = \cdots = a_k = 0$$

② $\operatorname{span}\beta = \bigoplus_{\alpha\in I} V_\alpha$:

Given any $y \in \bigoplus_{\alpha\in I} V_\alpha$,

$$y = y_{\alpha_1} + \cdots + y_{\alpha_k}$$

for some $y_{\alpha_i} \in V_{\alpha_i} = \operatorname{span}\beta_{\alpha_i}$

$$\Rightarrow \quad y_{\alpha_i} = a_1^{\alpha_i}\cdot x_1^{\alpha_i} + \cdots + a_{j_i}^{\alpha_i} x_{j_i}^{\alpha_i}$$

for some $a_1^{\alpha_i}, \ldots, a_{j_i}^{\alpha_i} \in \mathbb{k}$, $x_1^{\alpha_i}, \ldots, x_{j_i}^{\alpha_i} \in \beta_{\alpha_i} \subseteq \beta$

$$\Rightarrow \quad y = \sum_{i=1}^{k} \left(a_1^{\alpha_i} x_1^{\alpha_i} + \cdots + a_{j_i}^{\alpha_i} x_{j_i}^{\alpha_i}\right) \in \operatorname{span}\beta$$

$$\Rightarrow \quad \operatorname{span}\beta = \bigoplus_{\alpha\in I} V_\alpha \qquad \#$$

HW3.2

Def.

Let $f: W \to V$, $g: V \to \tilde{V}$ be linear maps.

Suppose $\ker f = 0 = \operatorname{im}(f_0)$

f is 1-1 . g is onto. $\Leftrightarrow \operatorname{im} g = \tilde{V}$

$= \ker(g)$

$\ker g = \operatorname{im} f$ $\leftrightarrow$ exact at V

Then we say

$$0 \xrightarrow{f_0} W \xrightarrow{f} V \xrightarrow{g} \tilde{V} \xrightarrow{g_0} 0$$

is a <u>short exact sequence</u>.

e.g. $W \leq V$, $\tilde{V} = V/W$

inclusion map   quotient map

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r, hook, "\iota"] & V \arrow[l, bend left, dashed, "p"] \arrow[r, two heads, "\pi"] & V/W \arrow[l, bend left, dashed, "j"] \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence because

ι is 1-1, π is onto

$\operatorname{im}(\iota) = W = \ker(\pi)$

(b) Show that $\exists$ linear $j : V/W \to V$ s.t. $\pi \circ j = id_{V/W}$

(c) Show that $\exists$ linear $p : V \longrightarrow W$ s.t. $p \circ \iota = id_W$

pf

Let β be a basis for $W \leq V$.

$\Rightarrow$ β is linearly independent in V.

$\Rightarrow$ $\exists\, \gamma$ which is a basis for V and

$\beta \subseteq \gamma$

Let

$p : V \to W$

be the linear map s.t.

$$p(x) = \begin{cases} \underline{\underline{x}}^{\in\beta} & \text{if } x \in \beta \\ 0 & \text{if } x \in \gamma - \beta \end{cases}$$

$\Rightarrow \quad im(p) = \operatorname{span}\beta = W$

So one can consider p as a linear map $V \to W$, and

$$(p \circ \iota)(x) = p(x) = x = id(x) \;\forall\, x \in \beta$$

$\Rightarrow \quad p \circ \iota = id_W$ —— (G)

(b) Let

$$\tilde{\beta} = \{ [x] \in V/W \mid x \in \gamma \setminus \beta \}$$

<u>Claim</u>

$\tilde{\beta}$ is a basis for V/W

pf

① $\operatorname{span}\tilde{\beta} = V/W$:

$\forall\; y \in V/W, \quad y = [z] = \pi(z)$ for some $z \in V$

$\Rightarrow \quad z = a_1x_1 + \cdots + a_nx_n$

for some $a_1, \ldots, a_n \in k$, $x_1, \ldots, x_n \in \gamma$

Suppose $x_1, \ldots, x_k \in \beta$

$$x_{k+1}, \dots, x_n \in \gamma \setminus \beta$$

Since $\pi(\beta) = 0$,

$$y = \pi(z) = \pi(\underbrace{a_1x_1 + \cdots + a_kx_k}_{\in W}) + \pi(a_{k+1}x_{k+1} + \cdots)$$

$$= a_{k+1}\underbrace{[x_{k+1}]}_{\in \tilde{\beta}} + \cdots + a_n\underbrace{[x_n]}_{\in \tilde{\beta}}$$

$$\in \operatorname{span} \tilde{\beta}$$

$$\Rightarrow \operatorname{span} \tilde{\beta} = V/W$$

② $\tilde{\beta}$ is linearly indep.

Suppose

$$a_1[x_1] + \cdots + a_k[x_k] = 0 \qquad , \ x_1, \dots, x_k \in \gamma \setminus \beta$$

$$\parallel$$

$$[a_1x_1 + \cdots + a_kx_k] = 0$$

$$\Rightarrow \quad a_1x_1 + \cdots + a_kx_k \in W$$

$$\Rightarrow \quad a_1 = \cdots = a_k = 0 \qquad \#$$

Let $j : V/W \to V$ be the linear map s.t.

$$j([x]) = x \qquad \forall x \in \gamma \setminus \beta$$

$$\Rightarrow (\pi \circ j)([x]) = \pi(x) = [x] \qquad \forall x \in \gamma \setminus \beta$$

$$= \mathrm{id}_{V/W}([x])$$

V/W ...

$+\ \tilde{\beta}$ is a basis

$\Rightarrow \quad \pi \circ j = id_{V/W}$ $\qquad\qquad \#$

(d)

$$\left\{ \begin{array}{l} p: V \to W, \\ p \circ \iota = id_W \end{array} \right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \ \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} \ \left\{ \begin{array}{l} j: V/W \to V \\ \pi \circ j = id_{V/W} \end{array} \right\}$$

① Given j, $\underline{\pi \circ j = id}$, define $p: V \to W$ as follows.

$\forall\, v \in V$, note

$$\underbrace{v - j\pi(v)}_{=\iota(y)\ \exists\, y \in W} \in \ker \pi = \operatorname{im} \iota$$

because

$$\pi(v - j\pi(v)) = \pi(v) - \boxed{\pi j}\,\pi(v)$$

$\boxed{\pi j} = id$

ι is 1-1

$$= \pi(v) - \pi(v) = 0$$

$\exists!\ \underbrace{p(v)}_{= y,\ = \psi(j)} \in W$ s.t.

$$\iota(p(v)) = v - j\pi(v)$$

Check $p: V \to W$ is linear:

$\iota(p(av_1 + bv_2)) = (av_1 + bv_2) - \dots$

$$\iota(p(av_1+bv_2)) = (av_1+bv_2) - j\pi(av_1+bv_2)$$

$$= a(v_1 - j\pi v_1) + b(v_2 - j\pi v_2)$$

$$= a\cdot \iota p(v_1) + b\cdot \iota p(v_2)$$

$$= \iota\,(ap(v_1) + bp(v_2))$$

ι is 1-1

$$\Rightarrow \quad p(av_1+bv_2) = ap(v_1) + bp(v_2)$$

$\Psi(j) = P$.

② Given $p: V \to W$, $p\iota = id$, define

$\bar{j} = \Phi(p)$ as follows.

$\bar{j}: V/W \to V$.

$\forall\ [x] \in V/W$,

$$\bar{j}([x]) = \underline{x - \iota p(x)} \quad \in V$$

Check:

(i) $\bar{j}$ is well-defined:

if $\underset{\pi(x)}{[x]} = [y] = \pi(y)$, then

$$\bar{j}([x]) - \bar{j}([y])$$

$$= x - y - \iota p(x-y) = 0 \,?$$

Note $\quad \pi(x) = \pi(y) \Rightarrow \pi(x-y) = 0$

$\Rightarrow \quad x-y \in \ker\pi = \operatorname{im}\iota$

$\Rightarrow \quad x-y = \iota(z)$ for some $z \in W$

So

$$\bar{j}([x]) - \bar{j}([y])$$

$$= \iota(z) - \iota \boxed{p(\iota}(z)) \quad \overset{id}{=}$$

$$= \iota(z) - \iota(z) = 0$$

(ii) $\bar{j}$ is linear

$$\bar{j}(a[x_1]+b[x_2])$$

$$= \bar{j}([ax_1+bx_2])$$

linear

$$= (ax_1+bx_2) - \underline{\iota p}(ax_1+bx_2)$$

$$= a(x_1 - \iota p x_1) + b(x_2 - \iota p x_2)$$

$$= a\,\bar{j}([x_1]) + b\,\bar{j}([x_2])$$

(iii) $\pi \circ \bar{j} = id_{V/W}$:

$$\pi(\bar{j}([x])) = \pi(x - \iota p x)$$

0 $\because$ $im(\iota) = \ker(\pi)$

$$= \pi(x) - \boxed{\pi\iota}p(x)$$

$$= \pi(x) = [x]$$

$$\phi(p) = j$$

③ $\phi \circ \psi = id$, $\quad \psi \circ \phi = id$

(i) Given $\underline{\underline{j}}$,

s.t. $\psi(j) = p$,

$\iota p(v) = v - j\pi v$. $\qquad x - j\pi x$

$\Rightarrow \quad \phi(p)([x]) = x - \boxed{\iota p x}$

$= j\pi(x) = \underline{\underline{j}}([x])$

So $\quad \phi \circ \psi = id$

(ii) Similarly, $\quad \psi \circ \phi = id$ $\qquad$ #

(e) Main observation:

Let $p = \psi(j)$

$\Rightarrow \forall x \in V$, $\qquad j(V/W)$

$$x = \underbrace{\iota p(x)}_{\in\, \iota(W)} + \overbrace{j\pi(x)}^{\in\, j(V/W)}$$

$$V \cong W \oplus V/W$$