Linear Algebra 10/9

<u>Recall</u> (Universal property for vec. sp.)

Let β be a set. $\Bbbk^{(\beta)}$

$$\begin{tikzcd}
x \in \beta \arrow[r, "\forall f"] \arrow[d, hook, "i"'] & W \\
1 \cdot x \quad \Bbbk^{(\beta)} \arrow[ur, dashed, "\exists! \text{ linear } T_f"'] &
\end{tikzcd}$$

map between sets

$\Rightarrow$

$$\left\{ \underset{\text{maps between sets}}{\beta \longrightarrow W} \right\} \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} \left\{ \underset{\text{linear maps}}{\Bbbk^{(\beta)} \longrightarrow W} \right\}$$

<u>Cor</u>

If V is another vector space with $j : \beta \longrightarrow V$ which satisfies the same property:

$$\beta \xrightarrow{\forall f} W$$

$$
\begin{tikzcd}
\beta \arrow[d, "j"'] & \\
V \arrow[r, dashed] & \exists!\, T_f \text{ linear}
\end{tikzcd}
$$

then $V \cong \Bbbk^{(\beta)}$

pf

① For $f = i : \beta \longrightarrow \Bbbk^{(\beta)} = W$,

$\exists!$ linear $T_i : V \longrightarrow \Bbbk^{(\beta)}$ s.t.

$$T_i \circ j = i$$

$$
\begin{tikzcd}
\beta \arrow[r, "i=f"] \arrow[d, "j"'] & \Bbbk^{(\beta)} \\
V \arrow[ur, dashed, "\exists!\, T_i"'] &
\end{tikzcd}
$$

② For $f = j : \beta \longrightarrow V$, $\exists!$ linear $T_j : \Bbbk^{(\beta)} \to W$

s.t.

$$T_j \circ i = j$$

$f = i$

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "j"] \arrow[d, "i"'] & V \\
\Bbbk^{(\mathcal{B})} \arrow[ur, dashed, "\exists!\, T_j"'] &
\end{tikzcd}$$

③

$$\begin{tikzcd}
& \mathcal{B} \arrow[d, "i"] \arrow[dr, "\bar{j}"] \arrow[drr, bend left, "\bar{i}"] & & \\
T_i \circ T_j : & \Bbbk^{(\mathcal{B})} \arrow[r, "T_{\bar{j}}"] \arrow[rr, bend right, "\mathrm{id}_{\Bbbk^{(\mathcal{B})}}"'] & V \arrow[r, "T_{\bar{i}}"] & \Bbbk^{(\mathcal{B})}
\end{tikzcd}$$

Note

$$(T_i \circ T_j) \circ \bar{i} = T_i \circ \bar{j} = \bar{i}$$

$$= \mathrm{id}_{\Bbbk^{(\mathcal{B})}} \circ \bar{i}$$

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\bar{i}"] \arrow[d, "\bar{i}"'] & \Bbbk^{(\mathcal{B})} \\
\Bbbk^{(\mathcal{B})} \arrow[ur, dashed, "\mathrm{id}_{\Bbbk^{(\mathcal{B})}}", "T_i \circ T_j"'] &
\end{tikzcd}$$

$\exists!$ linear map

$$\Rightarrow \quad T_i \circ T_j = \mathrm{id}_{\Bbbk^{(\mathcal{B})}}$$

Similarly,

$$T_j \circ T_i = \mathrm{id}_V$$

So

$$T_j : \mathbb{k}^{(B)} \to V$$

is an isomorphism #

§ Universal property for tensor products

Recall

$$V_1 \otimes \cdots \otimes V_n = \mathbb{k}^{(V_1 \times \cdots \times V_n)} \Big/ R_n$$

$$R_n := \mathrm{Span}\left\{ \begin{matrix} (v_1, \cdots, v_i + v_i', \cdots) - (v_1, \cdots, v_i, \cdots) - (v_1, \cdots, v_i', \cdots) \\ c \cdot (v_1, \cdots, v_n) - (v_1, \cdots, c \cdot v_i, \cdots, v_n) \end{matrix} \right\}$$

- $V_1 \otimes \cdots \otimes V_n = \mathrm{Span}\{ \vec{v}_1 \otimes \cdots \otimes \vec{v}_n \}$

  i.e.

  An element $x \in V_1 \otimes \cdots \otimes V_n$ is of the form

  $$x = \sum_{i=1}^{k} \vec{v}_1^{\,i} \otimes \cdots \otimes \vec{v}_n^{\,i}$$

- $\vec{v}_1 \otimes \cdots \otimes (\vec{v}_i + \vec{v}_i') \otimes \cdots \otimes \vec{v}_n$

- $\vec{v}_1 \otimes \cdots \otimes (\vec{v}_i + \vec{v}_i') \otimes \cdots \otimes \vec{v}_n$

$$= \vec{v}_1 \otimes \cdots \otimes \vec{v}_i \otimes \cdots \otimes \vec{v}_n + \vec{v}_1 \otimes \cdots \otimes \vec{v}_i' \otimes \cdots \otimes \vec{v}_n$$

- $\vec{v}_1 \otimes \cdots \otimes (c \cdot \vec{v}_i) \otimes \cdots \otimes \vec{v}_n$

$$= c \cdot (\vec{v}_1 \otimes \cdots \otimes \vec{v}_n)$$

e.g. In $\mathbb{R}^2 \otimes \mathbb{R}^2$, $\quad \vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$

$$(1,-1) \otimes (1,2)$$

$$= (\vec{e}_1 - \vec{e}_2) \otimes (\vec{e}_1 + 2\vec{e}_2)$$

$$= \vec{e}_1 \otimes \vec{e}_1 + \vec{e}_1 \otimes (2\vec{e}_2) + (-\vec{e}_2) \otimes \vec{e}_1 + (-\vec{e}_2) \otimes (2\vec{e}_2)$$

$$= \vec{e}_1 \otimes \vec{e}_1 + 2\, \vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1 - 2\, \vec{e}_2 \otimes \vec{e}_2$$

Note:

$$\vec{e}_1 \otimes \vec{e}_2 \neq \vec{e}_2 \otimes \vec{e}_1$$

Remark

The map

$$\varphi : V_1 \times \cdots \times V_n \longrightarrow V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

$$\varphi(\vec{v}_1, \cdots, \vec{v}_n) = \vec{v}_1 \otimes \cdots \otimes \vec{v}_n$$

is n-linear:

$$\underline{\varphi(\vec{v}_1, \cdots, a\vec{v}_i + b\vec{v}_i', \cdots, \vec{v}_n)}$$

$$= \vec{v}_1 \otimes \cdots \otimes (a\vec{v}_i + b\vec{v}_i') \otimes \cdots \otimes \vec{v}_n$$

$$= \vec{v}_1 \otimes \cdots \otimes (a\vec{v}_i) \otimes \cdots \otimes \vec{v}_n$$

$$+ \vec{v}_1 \otimes \cdots \otimes (b\vec{v}_i') \otimes \cdots \otimes \vec{v}_n$$

$$= a(\vec{v}_1 \otimes \cdots \vec{v}_i \cdots \otimes \vec{v}_n) + b(\vec{v}_1 \otimes \cdots \vec{v}_i' \cdots \otimes \vec{v}_n)$$

$$= \underline{a\varphi(\vec{v}_1, \cdots, \vec{v}_i, \cdots, \vec{v}_n) + b\varphi(\vec{v}_1, \cdots, \vec{v}_i', \cdots, \vec{v}_n)}$$

<u>Thm</u>

If $f : V_1 \times \cdots \times V_n \longrightarrow W$ is n-linear

then $\exists!$ linear map

$$\tilde{f} : V_1 \otimes \cdots \otimes V_n \longrightarrow W$$

s.t. $\frown$ $\tilde{\ }$

$$f' = f' \circ \varphi$$

$$\begin{tikzcd}
V_1 \times \cdots \times V_n \arrow[r, "\forall f", "n\text{-linear}"'] \arrow[d, "\varphi"'] & W \\
V_1 \otimes \cdots \otimes V_n \arrow[ur, dashed, "\exists!\, \tilde{f}"', "\text{linear}"] &
\end{tikzcd}$$

$\Rightarrow$

$$\left\{ \begin{matrix} V_1 \times \cdots \times V_n \longrightarrow W \\ n\text{-linear} \end{matrix} \right\} \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} \left\{ \begin{matrix} V_1 \otimes \cdots \otimes V_n \longrightarrow W \\ \text{linear} \end{matrix} \right\}$$

$$f \longmapsto \tilde{f}$$

<u>pf</u>

① Let $f : V_1 \times \cdots \times V_n \longrightarrow W$ be an n-linear map.

Then $\exists!$ linear map

$$T_f : \mathbb{k}^{(V_1 \times \cdots \times V_n)} \longrightarrow W$$

s.t.

$$(*) \quad T_f(\underbrace{1 \cdot (\vec{v}_1, \cdots, \vec{v}_n)}_{i(\vec{v}_1, \cdots, \vec{v}_n)}) = f(\vec{v}_1, \cdots, \vec{v}_n)$$

② Claim: $T_f(R_n) = \{0\}$

pf

Since f in n-linear, we have

(i) $T_f\left((\vec{v}_1, \dots, \vec{v}_i + \vec{v}_i', \dots, \vec{v}_n) - (\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) - (\vec{v}_1, \dots, \vec{v}_i', \dots, \vec{v}_n) \right)$

T_f is linear

$= T_f((\vec{v}_1, \dots, \vec{v}_i + \vec{v}_i', \dots, \vec{v}_n))$
$- T_f((\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n))$
$- T_f((\vec{v}_1, \dots, \vec{v}_i', \dots, \vec{v}_n))$

$\overset{(*)}{=} f(\vec{v}_1, \dots, \vec{v}_i + \vec{v}_i', \dots, \vec{v}_n)$
$- f(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n)$
$- f(\vec{v}_1, \dots, \vec{v}_i', \dots, \vec{v}_n)$

f is n-linear

$= 0$

(ii) $T_f\left((\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n) - c\cdot(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) \right)$

T_f is

$$\overset{\text{it is linear}}{=} T_f\left((\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n)\right)$$

$$- c \cdot T_f\left((\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n)\right)$$

$$\overset{\circledast}{=} f(\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n)$$

$$- c \cdot f(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n)$$

$$\underset{\text{n-linear}}{\overset{f \text{ is}}{=}} 0$$

So $T_f(R_n) = 0$

③ <u>Recall</u>

$$\begin{tikzcd}
\vec{v} \in V \arrow[r, "T"', "\text{linear}"] \arrow[d, twoheadrightarrow] & W \\
{[\vec{v}]} \in V/\tilde{V} \arrow[ur, dashed]
\end{tikzcd}$$

$\exists!$ linear $\tilde{T}$ s.t. $\tilde{T}([\vec{v}]) = T(\vec{v})$ $\Longleftrightarrow$ $T(\tilde{V}) = 0$

Since $T_f(R_n) = 0$,

$\exists$ linear map $\qquad V_1 \otimes \cdots \otimes V_n$

$$\tilde{f} : \frac{\mathbb{k}^{(V_1 \times \cdots \times V_n)}}{R_n} \longrightarrow W$$

$=$

s.t.

$$\begin{tikzcd}
x \in \mathbb{k}^{(V_1 \times \cdots \times V_n)} \arrow[r, "T_f"] \arrow[d] & W \\
\dfrac{\mathbb{k}^{(V_1 \times \cdots \times V_n)}}{R_n} \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

$[x]$

$$T_f(x) = f([x])$$

Note that

$$(\tilde{f} \circ \varphi) : V_1 \times \cdots \times V_n \xrightarrow{\varphi} V_1 \otimes \cdots \otimes V_n \xrightarrow{\tilde{f}} W$$

$$\underline{(\tilde{f} \circ \varphi)}(\vec{v}_1, \cdots, \vec{v}_n) = \tilde{f}\big(\varphi(\vec{v}_1, \cdots, \vec{v}_n)\big)$$

$$= \tilde{f}(\vec{v}_1 \otimes \cdots \otimes \vec{v}_n)$$

$$= \tilde{f}\big([1 \cdot (\vec{v}_1, \cdots, \vec{v}_n)]\big)$$

$$= T_f\big(1 \cdot (\vec{v}_1, \cdots, \vec{v}_n)\big)$$

$$= \underline{f}(\vec{v}_1, \cdots, \vec{v}_n)$$

So $$f = \tilde{f} \circ \varphi$$

④ Suppose
$$\tilde{g} : V_1 \otimes \cdots \otimes V_n \longrightarrow W$$
is another linear map s.t.
$$f = \tilde{g} \circ \varphi$$
$$(\overset{\shortparallel}{\tilde{f}} \circ \varphi)$$

$$\Rightarrow \tilde{f}(\varphi(\vec{v}_1, \ldots, \vec{v}_1)) = \tilde{f}(\vec{v}_1 \otimes \cdots \otimes \vec{v}_n)$$
$$= \tilde{g}(\vec{v}_1 \otimes \cdots \otimes \vec{v}_n)$$
$$\forall \ \vec{v}_i \in V_i, \quad i = 1, \ldots, n.$$

Since
$$V_1 \otimes \cdots \otimes V_n = \operatorname{span}\left\{ \vec{v}_1 \otimes \cdots \otimes \vec{v}_n \;\middle|\; \begin{matrix} \vec{v}_i \in V_i \\ i = 1, \ldots, n \end{matrix} \right\}$$
we have
$$\tilde{f} = \tilde{g}$$
#

$\forall x \in V_1 \otimes \cdots \otimes V_n,$

$$x = \sum_{j=1} \vec{v}_1^{\,j} \otimes \cdots \otimes \vec{v}_n^{\,j}$$

$$\Rightarrow \quad \tilde{f}(x) = \sum_{j=1}^{k} \tilde{f}(\vec{v}_1^{\,j} \otimes \cdots \otimes \vec{v}_n^{\,j})$$

$$= \sum_{j=1}^{k} \tilde{g}(\vec{v}_1^{\,j} \otimes \cdots \otimes \vec{v}_n^{\,j}) = \tilde{g}(x)$$

<u>Cor</u>

If $\tilde{V}$ is another vector space with n-linear map

$$\tilde{\varphi} : V_1 \times \cdots \times V_n \longrightarrow \tilde{V}$$

satisfying the universal property

$$\begin{tikzcd}
V_1 \times \cdots \times V_n \arrow[r, "\forall\ n\text{-linear}", "f"'] \arrow[d, "\tilde{\varphi}"'] & W \\
\tilde{V} \arrow[ur, dashed, "\exists!\ \text{linear}\ \tilde{f}"'] &
\end{tikzcd}$$

then $\tilde{V} \cong V_1 \otimes \cdots \otimes V_n$

pf: exer.

§ Universal properties for $\oplus$ and $\prod$

Let $\{V_\alpha\}_{\alpha \in I}$ be a collection of vector spaces

Let

$$\iota_\alpha : V_\alpha \hookrightarrow \bigoplus_{\alpha \in I} V_\alpha \quad \text{— natural inclusion}$$

$$\pi_\alpha : \prod_{\alpha \in I} V_\alpha \twoheadrightarrow V_\alpha \quad \text{— natural projection}$$

e.g.

$$\iota_1 : V_1 \longrightarrow V_1 \oplus V_2 \,, \quad \iota_1(x) = (x, 0)$$

$$\pi_1 : V_1 \oplus V_2 \longrightarrow V_1 \,, \quad \pi_1(x, y) = x$$

<u>Prop</u>

Let $f_\alpha : V_\alpha \longrightarrow W$, $g_\alpha : W \to V_\alpha$ be linear maps $\forall \alpha \in I$.

Then $\exists!$ linear maps

$$\tilde{f} : \bigoplus_{\alpha \in I} V_\alpha \to W \qquad \tilde{g} : W \to \prod_{\alpha \in I} V_\alpha$$

s.t.

$$\begin{tikzcd} V_\alpha \arrow[r, "f_\alpha"] \arrow[d, "\iota_\alpha"'] & W \\ \bigoplus_{\alpha\in I} V_\alpha \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "g_\alpha"] \arrow[dr, dashed, "\tilde{g}"'] & V_\alpha \\ & \prod_{\alpha\in I} V_\alpha \arrow[u, "\pi_\alpha"'] \end{tikzcd}$$

Example

Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$,

$$f_1(x) = x, \qquad f_2(x) = 2x$$

① $\mathbb{R} \oplus \mathbb{R} \xrightarrow{\tilde{f}} \mathbb{R}$

you know $\tilde{f}(x,0)$, $\tilde{f}(0,y)$ $\Leftrightarrow$ you know $\tilde{f}(x,y)$

$$\tilde{f}(x,y) = \tilde{f}(x,0) + \tilde{f}(0,y)$$

$$= f_1(x) + f_2(y) = x + 2y \in \mathbb{R}$$

② $\mathbb{R} \xrightarrow{\tilde{g}} \mathbb{R} \oplus \mathbb{R}$

$$x \longmapsto (f_1(x), f_2(x)) = (x, 2x)$$

§ Bases for tensor products

Prop

Let β be a basis for V

γ " " W

Then the set

$$S = \{ \vec{v} \otimes \vec{w} \mid \vec{v} \in \beta, \vec{w} \in \gamma \} \subseteq V \otimes W$$

is a basis for $V \otimes W$.

In particular,

$$\dim(V \otimes W) = \dim V \cdot \dim W.$$

pf

$\beta \times \gamma$ is a basis

Claim

$\mathbb{k}^{(\beta \times \gamma)} \cong V \otimes W$

Let $\mathbb{k}^{(\beta \times \gamma)}$ be the vector space freely generated by $\beta \times \gamma$

Let $\tilde{\varphi} : V \times W \to \mathbb{k}^{(\beta \times \gamma)}$ be the bilinear map

$$\tilde{\varphi}(\vec{v}, \vec{w}) = \sum_{i,j} a_i b_j (\vec{v}_i, \vec{w}_j)$$

where

$$\beta = \{ \vec{v}_i \mid i \in I \}, \quad \gamma = \{ \vec{w}_j \mid j \in J \}$$

$$\vec{v} = \sum a_i \vec{v}_i$$

$$\vec{w} = \sum b_j \vec{w}_j$$

exer: $\tilde{\varphi}$ is bilinear

Claim $(\mathbb{k}^{(\beta\times\gamma)}, \tilde{\varphi})$ satisfies the universal property for $V \otimes W$

pf

$$\begin{tikzcd} V \times W \arrow[r, "\forall f", "\text{bilinear}"'] \arrow[d, "\tilde{\varphi}"'] & W' \\ \mathbb{k}^{(\beta\times\gamma)} \arrow[ur, dashed, "\exists!\, \tilde{f}"', "\text{linear}"] & \end{tikzcd}$$

Goal

$\forall$ bilinear $f : V \times W \to W'$, since $\beta \times \gamma$ is a basis for $\mathbb{k}^{(\beta\times\gamma)}$, we have

$\exists!$ linear $\tilde{f} : \mathbb{k}^{(\beta\times\gamma)} \longrightarrow W'$

s.t.

$$\tilde{f}(1 \cdot (\vec{v}_i, \vec{w}_j)) = f(\vec{v}_i, \vec{w}_j)$$

For this f,

$$(\tilde{f}\circ\tilde{\varphi})(\vec{v},\vec{w}) = \tilde{f}\Big(\sum_{i,j} a_i b_j (\vec{v}_i,\vec{w}_j)\Big)$$

$$= \sum_{i,j} a_i b_j \underline{\tilde{f}(\vec{v}_i,\vec{w}_j)}$$

$$= \sum_{i,j} a_i b_j f(\vec{v}_i,\vec{w}_j)$$

$$= \sum_i a_i f\Big(\vec{v}_i,\ \overbrace{\sum b_j \vec{w}_j}^{\vec{w}}\Big)$$

$$= f\Big(\sum_i \vec{v}_i,\ \vec{w}\Big) = f(\vec{v},\vec{w})$$

So $\tilde{f}\circ\tilde{\varphi} = f$

Furthermore, if $\tilde{g}: \mathbb{k}^{(\beta\times\gamma)} \to W'$ is another linear map s.t.

$$\tilde{g}\circ\tilde{\varphi} = f$$

a basis for $\mathbb{k}^{(\beta\times\gamma)}$

then $(\tilde{g}\circ\tilde{\varphi})(\vec{v}_i,\vec{w}_j) = \tilde{g}(1\cdot\underline{(\vec{v}_i,\vec{w}_j)})$

$$= f(\vec{v}_i, \vec{w}_j) = \tilde{f}(1 \cdot (\vec{v}_i, \vec{w}_j))$$

Since $\tilde{g} = \tilde{f}$ on a basis,

one has $\tilde{g} = \tilde{f}$

Thus, by Cor,

$$V \otimes W \cong \mathbb{k}^{(\beta \times \gamma)}$$

To get an isomorphism,

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[d, "\tilde{\varphi}"'] & V \otimes W \\ \mathbb{k}^{(\beta \times \gamma)} \arrow[ur, dashed, "\exists ! \Phi \text{ linear}"'] & \end{tikzcd}$$

Here, Φ is the unique linear map s.t.

$$\Phi(1 \cdot (\vec{v}_i, \vec{w}_j)) = \varphi(\vec{v}_i, \vec{w}_j) = \vec{v}_i \otimes \vec{w}_j$$

By the proof of Cor, Φ is an

isomorphism $\quad \Phi: \mathbb{K}^{(\beta\times\gamma)} \to V\otimes W$

$$\Rightarrow \quad \Phi(\beta\times\gamma) = \{\vec{v}_i\otimes\vec{w}_j \mid \vec{v}_i\in\beta,\ \vec{w}_j\in\gamma\}$$
$$= S$$

is a basis for $V\otimes W$ #

<u>Example</u>

$\mathbb{R}^2\otimes\mathbb{R}^2$

Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$

<u>Prop</u>

$\Rightarrow$

$$\{\vec{e}_1\otimes\vec{e}_1,\ \vec{e}_1\otimes\vec{e}_2,\ \vec{e}_2\otimes\vec{e}_1,\ \vec{e}_2\otimes\vec{e}_2\}$$

is a basis for $\mathbb{R}^2\otimes\mathbb{R}^2$

So an element in $\mathbb{R}^2\otimes\mathbb{R}^2$ can be uniquely written as

$$a\,\vec{e}_1\otimes\vec{e}_1 + b\,\vec{e}_1\otimes\vec{e}_2 + c\,\vec{e}_2\otimes\vec{e}_1 + d\,\vec{e}_2\otimes\vec{e}_2$$

Recall the exercise:

$\vec{v} \otimes \vec{w} = 0 \iff \vec{v} = 0$ or $\vec{w} = 0$

Check it in $\mathbb{R}^2 \otimes \mathbb{R}^2$:

Let $\vec{v} = a\vec{e}_1 + b\vec{e}_2 \in \mathbb{R}^2$

$\vec{w} = c\vec{e}_1 + d\vec{e}_2 \in \mathbb{R}^2$

$$\Rightarrow \vec{v} \otimes \vec{w} = (a\vec{e}_1 + b\vec{e}_2) \otimes (c\vec{e}_1 + d\vec{e}_2)$$

$$= ac \cdot \vec{e}_1 \otimes \vec{e}_1 + ad \cdot \vec{e}_1 \otimes \vec{e}_2 + bc \cdot \vec{e}_2 \otimes \vec{e}_1 + bd \cdot \vec{e}_2 \otimes \vec{e}_2$$

$$= 0$$

$$\iff ac = ad = bc = bd = 0$$

Note:

$ac = 0 \Rightarrow a = 0$ or $c = 0$

$bd = 0 \Rightarrow b = 0$ or $d = 0$

If $a = 0$, $b = 0$, then $\vec{v} = 0$

If $a = 0$, $d = 0$, then

$$bc = 0 \Rightarrow \underset{\Downarrow \atop \vec{v} = 0}{\underline{b = 0}} \quad \text{or} \quad \underset{\Downarrow \atop \vec{w} = 0}{\underline{c = 0}}$$

$\therefore$

$$\Rightarrow \vec{v} = 0 \quad \text{or} \quad \vec{w} = 0 \qquad \#$$