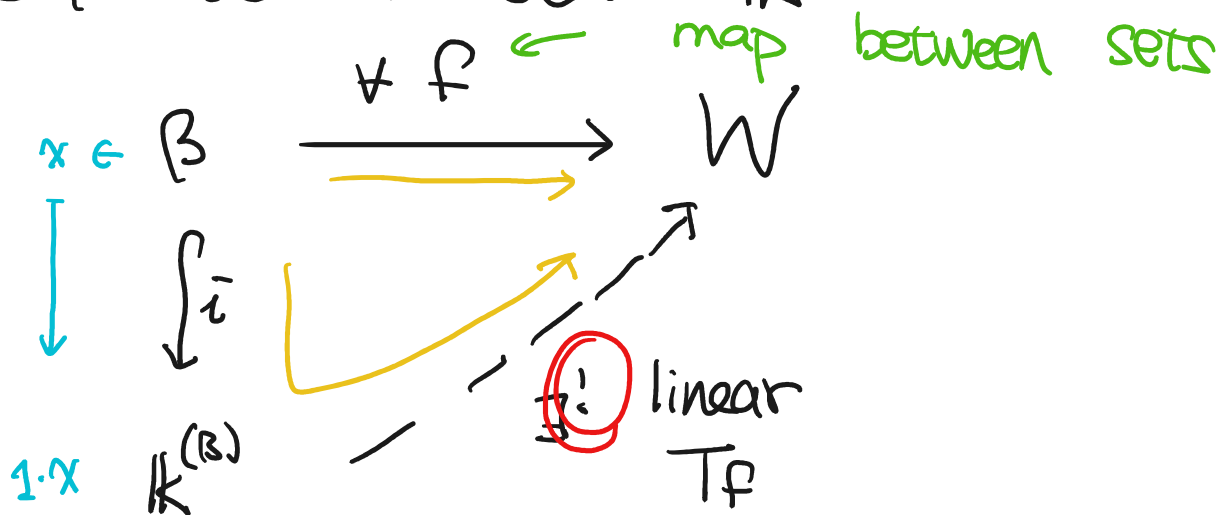


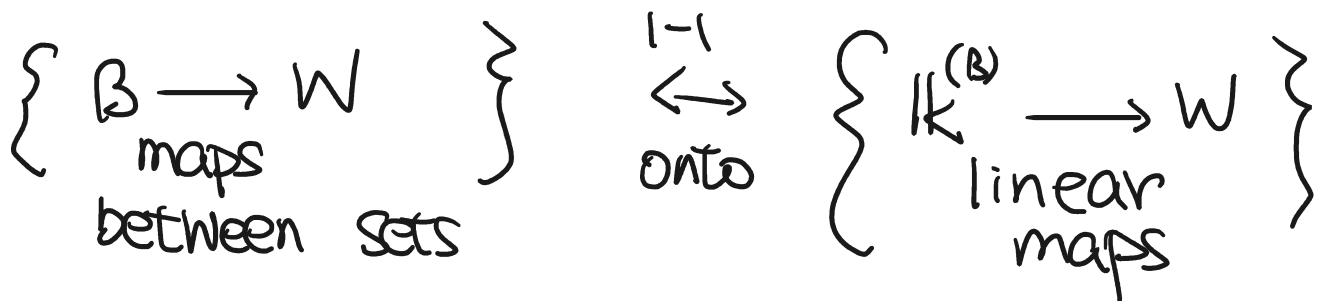
Linear Algebra 10/19

Recall (Universal property for vec sp.)

Let B be a set. $K^{(B)}$



\Rightarrow



Cor

If V is another vector space with $j: B \rightarrow V$ which satisfies the same property:

$$B \xrightarrow{\forall f} W$$



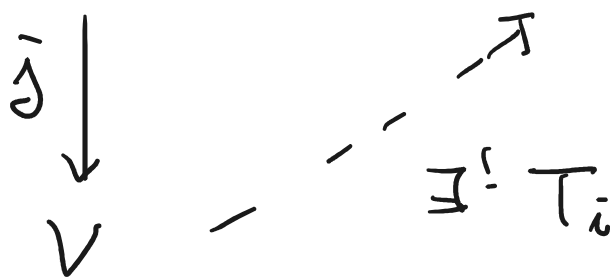
then

$$V \cong K^{(B)}$$

① $f = i : B \rightarrow K^{(B)} = W$
 \exists linear $T_i : V \rightarrow K^{(B)}$ s.t.

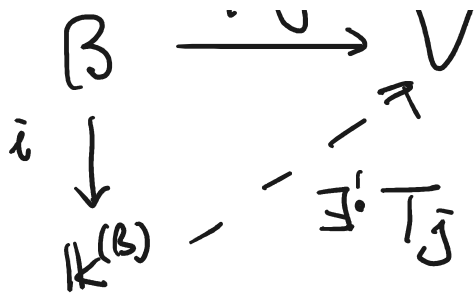
$$T_i \circ \tilde{j} = \tilde{i}$$

$$B \xrightarrow{\tilde{i} = f} K^{(B)}$$

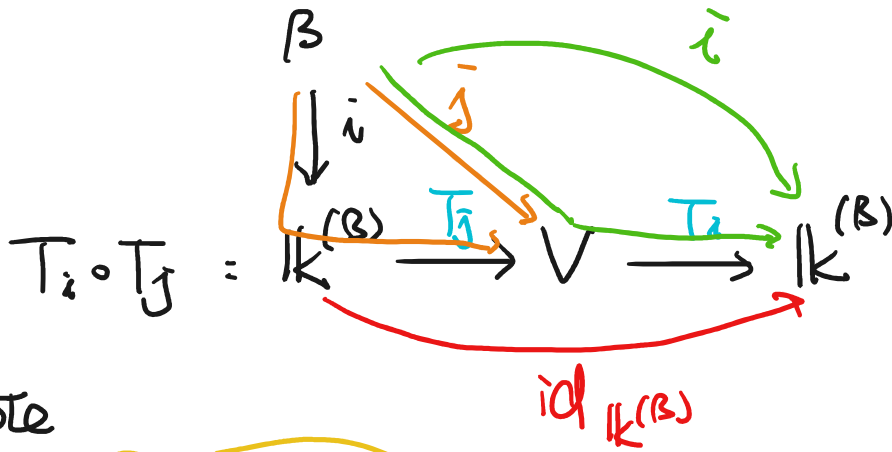


② For $f = \tilde{j} : B \rightarrow V$, \exists linear $T_j : K^{(B)} \rightarrow W$
 s.t.

$$T_j \circ \tilde{i} = \tilde{j}$$

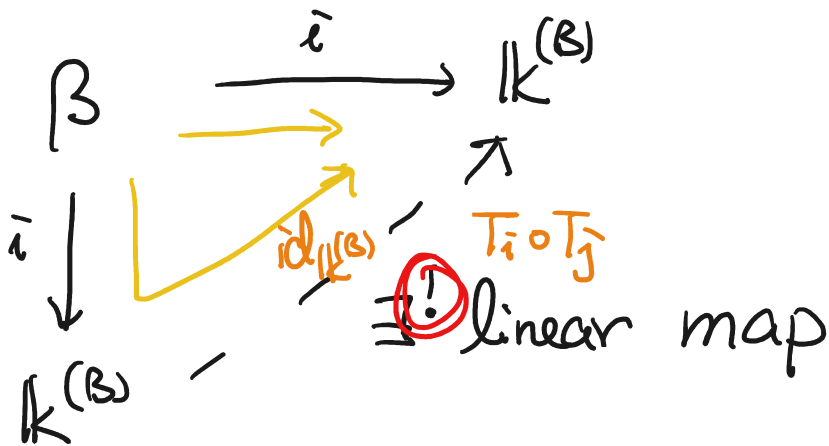


③



Note

$$\begin{aligned}
 (T_i \circ T_j) \circ i &= T_i \circ j = i \\
 &= \text{id}_{K^{(\beta)}} \circ i
 \end{aligned}$$



$$\Rightarrow T_i \circ T_j = \text{id}_{K^{(\beta)}}$$

Similarly,

$$T_j \circ T_i = \text{id}_V$$

So

$$T_j: K^{(R)} \rightarrow V$$

is an isomorphism $\#$

§ Universal property for tensor products

Recall

$$V_1 \otimes \dots \otimes V_n = K^{(V_1 \times \dots \times V_n)}$$

$R_n :=$

$$\text{Span} \left\{ \begin{aligned} &(v_1, \dots, v_i + v_i', \dots) \\ &- (v_1, \dots, v_i, \dots) - (v_1, \dots, v_i', \dots) \\ &c \cdot (v_1, \dots, v_n) \\ &- (c \cdot v_1, \dots, c \cdot v_i, \dots, v_n) \end{aligned} \right\}$$

- $V_1 \otimes \dots \otimes V_n = \text{Span} \{ \vec{v}_1 \otimes \dots \otimes \vec{v}_n \}$

i.e.

An element $x \in V_1 \otimes \dots \otimes V_n$ is of the form

$$x = \sum_{i=1}^k \vec{v}_1^i \otimes \dots \otimes \vec{v}_n^i$$

- $\vec{v}_1 \otimes \dots \otimes (c \cdot \vec{v}_i) \otimes \dots \otimes \vec{v}_n$

$$\begin{aligned} & v_1 \otimes \dots \otimes (v_i + v_i') \otimes \dots \otimes v_n \\ &= \underbrace{v_1 \otimes \dots \otimes v_i}_{\text{blue}} \otimes \dots \otimes v_n + \underbrace{v_1 \otimes \dots \otimes v_i'}_{\text{blue}} \otimes \dots \otimes v_n \end{aligned}$$

$$\begin{aligned} & v_1 \otimes \dots \otimes (c \cdot v_i) \otimes \dots \otimes v_n \\ &= c \cdot (v_1 \otimes \dots \otimes v_n) \end{aligned}$$

eg. In $\mathbb{R}^2 \otimes \mathbb{R}^2$, $\vec{e}_1 = (1, 0)$, $\vec{e}_2 = (0, 1)$

$$(1, -1) \otimes (1, 2)$$

$$= (\vec{e}_1 - \vec{e}_2) \otimes (\vec{e}_1 + 2\vec{e}_2)$$

$$\begin{aligned} &= \vec{e}_1 \otimes \vec{e}_1 + \vec{e}_1 \otimes (2\vec{e}_2) + (-\vec{e}_2) \otimes \vec{e}_1 \\ &\quad + (-\vec{e}_2) \otimes (2\vec{e}_2) \end{aligned}$$

$$= \vec{e}_1 \otimes \vec{e}_1 + 2 \vec{e}_1 \otimes \vec{e}_2$$

$$- \vec{e}_2 \otimes \vec{e}_1 - 2 \vec{e}_2 \otimes \vec{e}_2$$

Note:

$$\vec{e}_1 \otimes \vec{e}_2 \neq \vec{e}_2 \otimes \vec{e}_1$$

Remark

The map

$$\varphi: V_1 \times \dots \times V_n \rightarrow V_1 \otimes V_2 \otimes \dots \otimes V_n$$

$$\varphi(\vec{v}_1, \dots, \vec{v}_n) = \vec{v}_1 \otimes \dots \otimes \vec{v}_n$$

is n -linear:

$$\varphi(\vec{v}_1, \dots, a\vec{v}_i + b\vec{v}_i', \dots, \vec{v}_n)$$

$$= \vec{v}_1 \otimes \dots \otimes (a\vec{v}_i + b\vec{v}_i') \otimes \dots \otimes \vec{v}_n$$

$$= \vec{v}_1 \otimes \dots \otimes (a\vec{v}_i) \otimes \dots \otimes \vec{v}_n$$

$$+ \vec{v}_1 \otimes \dots \otimes (b\vec{v}_i') \otimes \dots \otimes \vec{v}_n$$

$$= a(\vec{v}_1 \otimes \dots \otimes \vec{v}_i \otimes \dots \otimes \vec{v}_n) + b(\vec{v}_1 \otimes \dots \otimes \vec{v}_i' \otimes \dots \otimes \vec{v}_n)$$

$$= a\varphi(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) + b\varphi(\vec{v}_1, \dots, \vec{v}_i', \dots, \vec{v}_n)$$

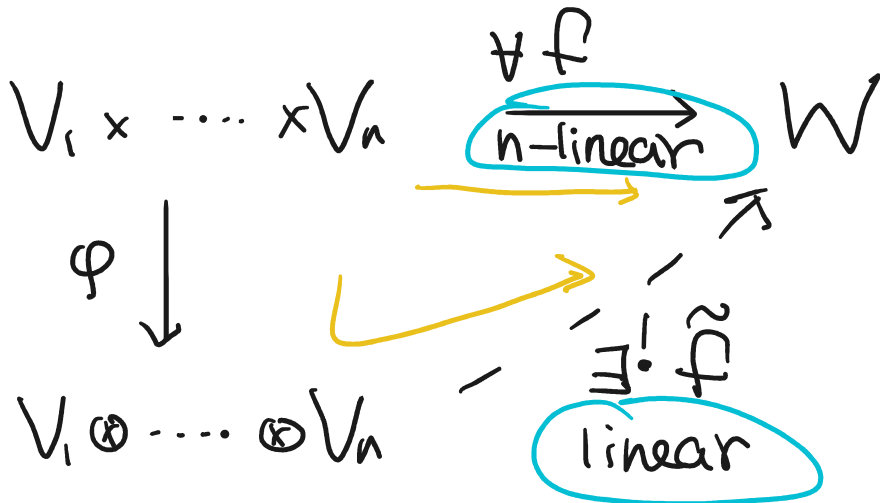
Thm

If $f: V_1 \times \dots \times V_n \rightarrow W$ is n -linear,
then $\exists!$ linear map

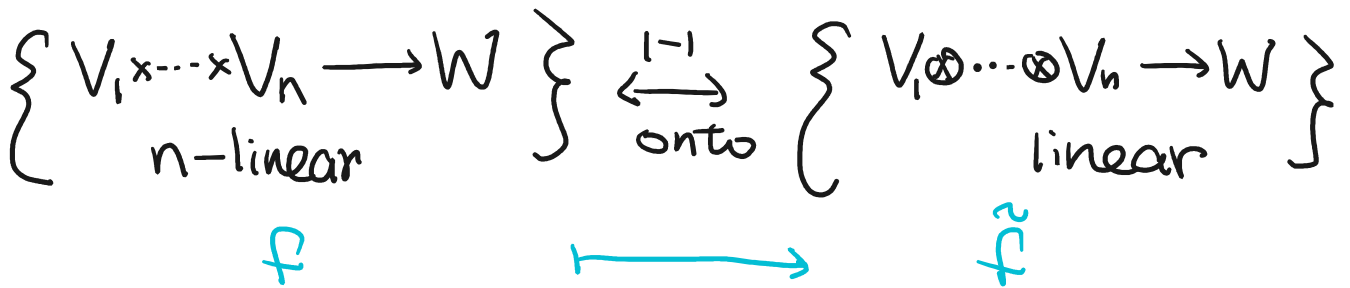
$$\tilde{f}: V_1 \otimes \dots \otimes V_n \rightarrow W$$

s.t. $\tilde{f} \circ \varphi = f$

$$f' = f \circ \varphi$$



\Rightarrow



pf

① Let $f: V_1 \times \dots \times V_n \rightarrow W$ be an n -linear map.

Then \exists linear map

$$T_f: K^{(V_1 \times \dots \times V_n)} \longrightarrow W$$

s.t.

$$\circledast \quad T_f(\underbrace{1 \cdot (\vec{v}_1, \dots, \vec{v}_n)}_{i(\vec{v}_1, \dots, \vec{v}_n)}) = f(\vec{v}_1, \dots, \vec{v}_n)$$

② Claim: $T_f(R_n) = \{0\}$

~~f~~

Since f is n -linear, we have

$$\text{ii) } T_f \left((\vec{v}_1, \dots, \vec{v}_i + \vec{v}_i', \dots, \vec{v}_n) - (\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) \right. \\ \left. - (\vec{v}_1, \dots, \vec{v}_i', \dots, \vec{v}_n) \right)$$

T_f is
linear

$$\begin{aligned} &= T_f(\vec{v}_1, \dots, \vec{v}_i + \vec{v}_i', \dots, \vec{v}_n) \\ &\quad - T_f(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) \\ &\quad - T_f(\vec{v}_1, \dots, \vec{v}_i', \dots, \vec{v}_n) \end{aligned}$$

$$\begin{aligned} &\stackrel{(*)}{=} f(\vec{v}_1, \dots, \underline{\vec{v}_i + \vec{v}_i'}, \dots, \vec{v}_n) \\ &\quad - f(\vec{v}_1, \dots, \underline{\vec{v}_i}, \dots, \vec{v}_n) \\ &\quad - f(\vec{v}_1, \dots, \underline{\vec{v}_i'}, \dots, \vec{v}_n) \end{aligned}$$

f is

n -linear $= 0$

$$\text{iii) } T_f \left((\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n) - c \cdot (\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) \right)$$

T_f is

$$\stackrel{\text{if } f \text{ is}}{\text{linear}} = T_f(c\vec{v}_1, \dots, c\vec{v}_i, \dots, \vec{v}_n) \\ = c \cdot T_f(c\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n)$$

$$\stackrel{\text{if } f \text{ is}}{=} f(\vec{v}_1, \dots, \underline{c\vec{v}_i}, \dots, \vec{v}_n) \\ = \underline{c} \cdot f(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n)$$

$$f \text{ is } \\ = \text{O} \\ \text{n-linear}$$

$$\text{So } \underline{T_f(R_n) = \text{O}}$$

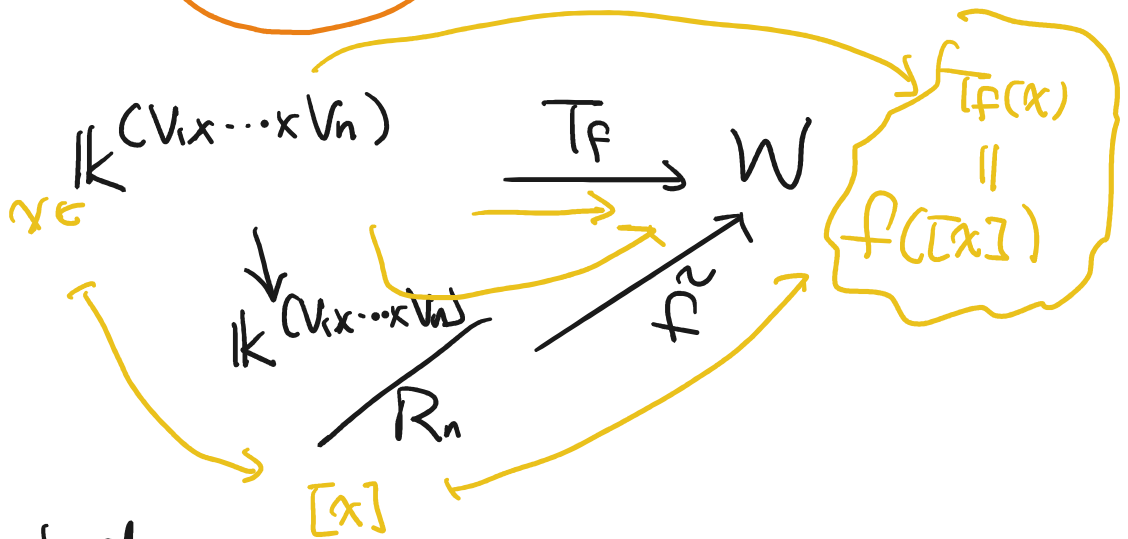
③ Recall

$$\begin{array}{ccc} \vec{v} \in V & \xrightarrow[\text{linear}]{T} & W \\ \downarrow & \searrow & \\ \vec{v} \in V & \xrightarrow{\tilde{T}} & \tilde{W} \end{array} \quad \begin{array}{l} \exists \text{ linear } \\ \tilde{T} \\ \text{s.t. } \tilde{T}(\vec{v}) = T(\vec{v}) \end{array} \Leftrightarrow T(\tilde{V}) = \text{O}$$

$$\text{Since } T_f(R_n) = \text{O},$$

\exists linear map
 $f_{\tilde{2}}: \frac{\mathbb{K}(V_1 \times \dots \times V_n)}{R_n} \rightarrow W$
 $= V_1 \otimes \dots \otimes V_n$

sit.



Note that

$$(\tilde{f} \circ \varphi) : V_1 \times \dots \times V_n \xrightarrow{\varphi} V_1 \otimes \dots \otimes V_n \xrightarrow{\tilde{f}} W$$

$$\underline{(\tilde{f} \circ \varphi)}(\vec{v}_1, \dots, \vec{v}_n) = \tilde{f}(\varphi(\vec{v}_1, \dots, \vec{v}_n))$$

$$= \tilde{f}(\vec{v}_1 \otimes \dots \otimes \vec{v}_n)$$

$$= \tilde{f}([1 \cdot (\vec{v}_1, \dots, \vec{v}_n)])$$

$$= T_f(1 \cdot (\vec{v}_1, \dots, \vec{v}_n))$$

$$= \underline{f}(\vec{v}_1, \dots, \vec{v}_n)$$

So $f = \tilde{f} \circ \varphi$

④ Suppose

$$\tilde{g}: V_1 \otimes \dots \otimes V_n \rightarrow W$$

is another linear map s.t.

$$f = \tilde{g} \circ \varphi$$

($\tilde{f} \circ \varphi$)

$$\Rightarrow \tilde{f}(\varphi(\vec{v}_1, \dots, \vec{v}_n)) = \tilde{f}(\vec{v}_1 \otimes \dots \otimes \vec{v}_n) \\ = \tilde{g}(\vec{v}_1 \otimes \dots \otimes \vec{v}_n)$$

$$\forall \vec{v}_i \in V_i, \quad i=1, \dots, n.$$

Since

$$V_1 \otimes \dots \otimes V_n = \text{span} \left\{ \vec{v}_1 \otimes \dots \otimes \vec{v}_n \mid \vec{v}_i \in V_i, \quad i=1, \dots, n \right\}$$

we have

$$\tilde{f} = \tilde{g}$$

#

$$\forall x \in V_1 \otimes \dots \otimes V_n,$$

$\leftarrow \dots \rightarrow$

$$x = \sum_{j=1}^r \vec{v}_1^{j_1} \otimes \dots \otimes \vec{v}_n^{j_n}$$

$$\begin{aligned} \Rightarrow \tilde{f}(x) &= \sum_{j=1}^r \tilde{f}(\vec{v}_1^{j_1} \otimes \dots \otimes \vec{v}_n^{j_n}) \\ &= \sum_{j=1}^r \tilde{g}(\vec{v}_1^{j_1} \otimes \dots \otimes \vec{v}_n^{j_n}) = \tilde{g}(x) \end{aligned}$$

Cor

If \tilde{V} is another vector space with n -linear map

$$\tilde{\varphi} : V_1 \times \dots \times V_n \longrightarrow \tilde{V}$$

satisfying the universal property

$$\begin{array}{ccc} V_1 \times \dots \times V_n & \xrightarrow{\text{A } n\text{-linear } f} & W \\ \tilde{\varphi} \downarrow & \nearrow \text{linear } \tilde{f} & \uparrow \\ \tilde{V} & & \end{array}$$

then $\tilde{V} \cong V_1 \otimes \dots \otimes V_n$

pf: exer.

§ Universal properties for \oplus and \prod

Let $\{V_\alpha\}_{\alpha \in I}$ be a collection of vector spaces

Let

$$\iota_\alpha: V_\alpha \hookrightarrow \bigoplus_{\alpha \in I} V_\alpha \quad \text{— natural inclusion}$$

$$\pi_\alpha: \prod_{\alpha \in I} V_\alpha \twoheadrightarrow V_\alpha \quad \text{— natural projection}$$

eg.

$$\iota_1: V_1 \longrightarrow V_1 \oplus V_2, \quad \iota_1(x) = (x, 0)$$

$$\pi_1: V_1 \oplus V_2 \longrightarrow V_1, \quad \pi_1(x, y) = x$$

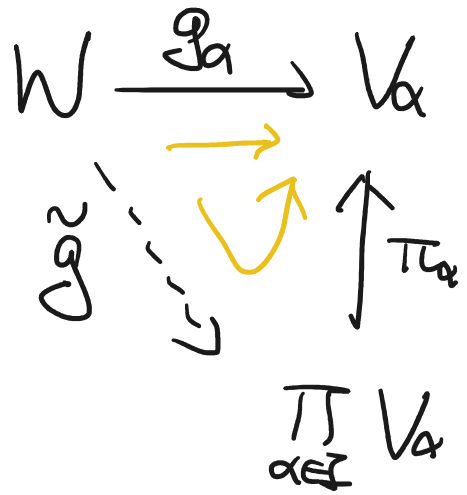
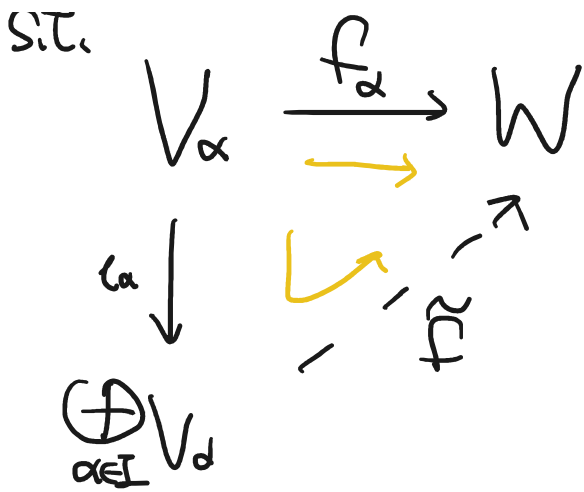
Prop

Let $f_\alpha: V_\alpha \rightarrow W$, $g_\alpha: W \rightarrow V_\alpha$
be linear maps $\forall \alpha \in I$.

Then $\exists!$ linear maps

$$\hat{f}: \bigoplus_{\alpha \in I} V_\alpha \rightarrow W$$

$$\hat{g}: W \rightarrow \prod_{\alpha \in I} V_\alpha$$



Example

Let $f_1, f_2: \mathbb{R} \rightarrow \mathbb{R}$.

$$f_1(x) = x, \quad f_2(x) = 2x$$

①

$$\begin{array}{ccc}
 \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\
 \downarrow & \searrow & \downarrow \\
 \mathbb{R} \oplus \mathbb{R} & \xrightarrow{f_2} & \mathbb{R}
 \end{array}$$

you know $\hat{f}(x,0)$
 \Leftrightarrow you know $\hat{f}(0,y)$

$$\begin{aligned}
 \hat{f}(x,y) &= \hat{f}(x,0) + \hat{f}(0,y) \\
 &= f_1(x) + f_2(y) = x + 2y \in \mathbb{R}
 \end{aligned}$$

②

$$\begin{array}{ccc}
 \mathbb{R} & \xrightarrow{g} & \mathbb{R} \oplus \mathbb{R} \\
 x & \longmapsto & (f_1(x), f_2(x)) = (x, 2x)
 \end{array}$$

§ Bases for tensor products

Prop

Let β be a basis for V
 γ " " " W

Then the set

$$S = \{ \vec{v} \otimes \vec{w} \mid \vec{v} \in \beta, \vec{w} \in \gamma \} \subseteq V \otimes W$$

is a basis for $V \otimes W$.

In particular,

$$\dim(V \otimes W) = \dim V \cdot \dim W.$$

pf

$\beta \times \gamma$ is a basis

Claim

$$k^{(\beta \times \gamma)} \cong V \otimes W$$

Let $k^{(\beta \times \gamma)}$ be the vector space
freely generated by $\beta \times \gamma$

Let $\tilde{\varphi} : V \times W \rightarrow k^{(\beta \times \gamma)}$ be the
bilinear map

$$\tilde{\varphi}(\vec{v}, \vec{w}) = \sum_i a_i b_i (\vec{v}_i, \vec{w}_i)$$

where

$$\beta = \{ \vec{v}_i \mid i \in I \}, \quad \gamma = \{ \vec{w}_j \mid j \in J \}$$

$$\vec{v} = \sum a_i \vec{v}_i$$

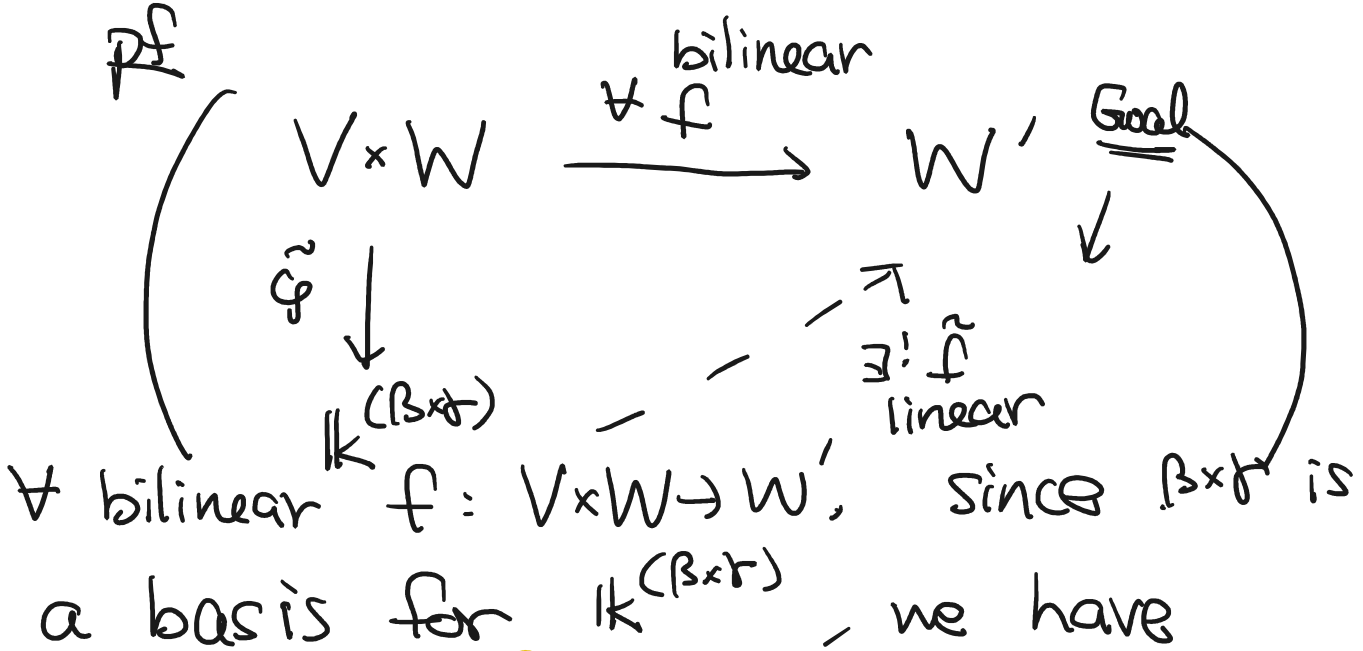
$$\vec{w} = \sum b_j \vec{w}_j$$

exer: $\tilde{\varphi}$ is bilinear

Claim $(K^{(\beta \times \gamma)}, \tilde{\varphi})$ satisfies

the universal property for $V \otimes W$

pf



$$\exists! \text{ linear } \tilde{f}: K^{(\beta \times \gamma)} \rightarrow W'$$

s.t.

$$\tilde{f}(1 \cdot (\vec{v}_i, \vec{w}_j)) = f(\vec{v}_i, \vec{w}_j)$$

for this f ,

$$(f \circ \tilde{\varphi})(\vec{v}, \vec{w}) = \tilde{f}\left(\sum_{i,j} a_i b_j (\vec{v}_i, \vec{w}_j)\right)$$

$$= \sum_{i,j} a_i b_j \underline{\tilde{f}(\vec{v}_i, \vec{w}_j)}$$

$$= \sum_{i,j} a_i b_j f(\vec{v}_i, \vec{w}_j)$$

$$= \sum_i a_i f(\vec{v}_i, \underbrace{\sum_j b_j \vec{w}_j}_{\vec{w}})$$

$$= f\left(\sum_i \vec{v}_i, \vec{w}\right) = f(\vec{v}, \vec{w})$$

So $\tilde{f} \circ \tilde{\varphi} = f$

Furthermore, if $\tilde{g}: \mathbb{K}^{(B \times D)} \rightarrow W'$
is another linear map s.t.

$$\tilde{g} \circ \tilde{\varphi} = f$$

then

$$(\tilde{g} \circ \tilde{\varphi})(\vec{v}_i, \vec{w}_j) = \tilde{g}(1 \cdot \underline{\vec{v}_i, \vec{w}_j})$$

a basis
for $\mathbb{K}^{(B \times D)}$

$$= f(\vec{v}_i, \vec{w}_j) = \tilde{f}(1 \cdot (\vec{v}_i, \vec{w}_j))$$

Since $\tilde{g} = \tilde{f}$ on a basis,

one has $\tilde{g} = \tilde{f}$

Thus, by Cor.,

$$V \otimes W \cong K^{(B \times \mathcal{B})}$$

To get an isomorphism,

$$\begin{array}{ccc} V \times W & \xrightarrow{\varphi} & V \otimes W \\ \downarrow \cong & \searrow & \uparrow \\ K^{(B \times \mathcal{B})} & & \end{array}$$

Here, $\bar{\Phi}$ is the unique linear map s.t.

$$\bar{\Phi}(1 \cdot (\vec{v}_i, \vec{w}_j)) = \varphi(\vec{v}_i, \vec{w}_j) = \vec{v}_i \otimes \vec{w}_j$$

By the proof of Cor., $\bar{\Phi}$ is an

isomorphism

$$\bar{\Phi}: K^{(\beta \times \gamma)} \rightarrow V \otimes W$$

$$\Rightarrow \bar{\Phi}(\beta \times \gamma) = \{ \vec{v}_i \otimes \vec{w}_j \mid \vec{v}_i \in \beta, \vec{w}_j \in \gamma \}$$

$$\cong S$$

is a basis for $V \otimes W$ $\#$

Example

$$\mathbb{R}^2 \otimes \mathbb{R}^2$$

Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for \mathbb{R}^2

Prop
 \Rightarrow

$$\{\vec{e}_1 \otimes \vec{e}_1, \vec{e}_1 \otimes \vec{e}_2, \vec{e}_2 \otimes \vec{e}_1, \vec{e}_2 \otimes \vec{e}_2\}$$

is a basis for $\mathbb{R}^2 \otimes \mathbb{R}^2$

So an element in $\mathbb{R}^2 \otimes \mathbb{R}^2$ can be uniquely written as

$$a \vec{e}_1 \otimes \vec{e}_1 + b \vec{e}_1 \otimes \vec{e}_2 + c \vec{e}_2 \otimes \vec{e}_1 + d \vec{e}_2 \otimes \vec{e}_2$$

Recall the exercise:

$$\vec{v} \otimes \vec{w} = 0 \Leftrightarrow \vec{v} = 0 \text{ or } \vec{w} = 0$$

Check it in $\mathbb{R}^2 \otimes \mathbb{R}^2$:

$$\text{Let } \vec{v} = a\vec{e}_1 + b\vec{e}_2 \in \mathbb{R}^2$$

$$\vec{w} = c\vec{e}_1 + d\vec{e}_2 \in \mathbb{R}^2$$

$$\Rightarrow \vec{v} \otimes \vec{w} = (a\vec{e}_1 + b\vec{e}_2) \otimes (c\vec{e}_1 + d\vec{e}_2)$$

$$= ac \cdot \underline{\underline{\vec{e}_1 \otimes \vec{e}_1}} + ad \cdot \underline{\underline{\vec{e}_1 \otimes \vec{e}_2}}$$

$$+ bc \cdot \underline{\underline{\vec{e}_2 \otimes \vec{e}_1}} + bd \cdot \underline{\underline{\vec{e}_2 \otimes \vec{e}_2}}$$

$$= 0$$

$$\Leftrightarrow \underline{ac} = \underline{ad} = \underline{bc} = \underline{bd} = 0$$

Note:

$$ac = 0 \Rightarrow a = 0 \text{ or } c = 0$$

$$bd = 0 \Rightarrow b = 0 \text{ or } d = 0$$

If $a = 0$, $b = 0$, then $\vec{v} = 0$

If $a = 0$, $d = 0$, then

$$bc = 0 \Rightarrow \underline{b=0} \text{ or } \underline{c=0}$$

$\begin{matrix} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{matrix}$ $\begin{matrix} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{matrix}$

...

\Rightarrow

$$\underline{v=0}$$

or

$$\underline{w=0}$$

#