Linear Algebra 10/12

Let us start with a set $\beta = \{\spadesuit, \clubsuit\}$ of 2 elements. The vector space $\Bbbk^{(\beta)}$ is the space

$$\Bbbk^{(\beta)} := \{a\spadesuit + b\clubsuit \mid a, b \in \Bbbk\}$$

which follows the rules:

(1) $a\spadesuit + b\clubsuit = a'\spadesuit + b'\clubsuit$ if and only if $a = a'$ and $b = b'$;
(2) $c(a\spadesuit + b\clubsuit) + d(a'\spadesuit + b'\clubsuit) = (ca + da')\spadesuit + (cb + db')\clubsuit$.

Note that the map $\Bbbk^2 \to \Bbbk^{(\beta)} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto a\spadesuit + b\clubsuit$ is an isomorphism, and thus $\beta = \{\spadesuit, \clubsuit\}$ is a basis for $\Bbbk^{(\beta)}$.

More generally, let β be an arbitrary set. The vector space $\Bbbk^{(\beta)}$ is the space

$$\begin{aligned}\Bbbk^{(\beta)} &:= \{\text{all the maps } \beta \to \Bbbk,\ x \mapsto a_x \text{ such that } a_x = 0 \text{ except for finitely many elements } x \in \beta\} \\ &= \{\textstyle\sum a_x\, x \mid a_x = 0, \text{ except for finitely many elements } x \in \beta\} \quad \text{(a common notation)}\end{aligned}$$

which follows the rules:

(1) $\sum a_x\, x = \sum b_x\, x$ if and only if $a_x = b_x$ for any $x \in \beta$;
(2) $c(\sum a_x\, x) + d(\sum b_x\, x) = \sum(ca_x + db_x)\, x$.

The map $\beta \to \Bbbk^{(\beta)}$, $x \mapsto 1_{\Bbbk}\, x$ is a one-to-one map, so β can be regarded as a subset of $\Bbbk^{(\beta)}$. It is straightforward to show that following

Construction of $V \otimes W$:

(V, W : $\Bbbk$-vector spaces)

Step 1

We consider $V \times W$ as a set and generate a vector space

$$\Bbbk^{(V \times W)} = \left\{ \sum_{(v,w) \in V \times W} a_{(v,w)} \cdot (v, w) \;\middle|\; \begin{array}{l} a_{(v,w)} \in \Bbbk, \\ a_{(v,w)} = 0 \\ \text{except finite} \end{array} \right\}$$

Note $V\times W$ is a basis for $\mathbb{k}^{(V\times W)}$

$(v,w) \in V\times W$]

<u>Step 2</u>

R = the subspace of $\mathbb{k}^{(V\times W)}$ spanned by the vectors of one of the forms

$$(v_1+v_2, w) - (v_1, w) - (v_2, w)$$
$$(v, w_1+w_2) - (v, w_1) - (v, w_2)$$
$$(c\cdot v, w) - c\cdot(v, w)$$
$$(v, c\cdot w) - c\cdot(v, w)$$

where

$v, v_1, v_2 \in V$, $\quad w, w_1, w_2 \in W$, $\quad c \in \mathbb{k}$

<u>Step 3</u>

The tensor product $V \otimes_{\mathbb{k}} W$

(or simply $V\otimes W$) is

$(V\times W)$ /

$$V \otimes W = \mathbb{k}^{(V \times W)} \big/ R$$

Notation

$$\vec{v} \otimes \vec{w} = [(\vec{v}, \vec{w})] \quad \text{for } \begin{array}{l} \vec{v} \in V \\ \vec{w} \in W \end{array}$$

An element $x \in V \otimes W$ of the form $\vec{v} \otimes \vec{w}$ is called a simple tensor

Remark

Let $x \in V \otimes W$. Then $\exists$

$$\vec{v}_1, \cdots, \vec{v}_k \in V, \quad \vec{w}_1, \cdots, \vec{w}_k \in W$$

s.t.

$$x = \underbrace{\sum_{i=1}^{k} a_i \cdot \vec{v}_i \otimes \vec{w}_i}_{\mathbin{/\!/}}$$

a general form of elements in $V \otimes W$

$$\mathbb{k}^{(V \times W)} \big/ R$$

$$\Rightarrow \quad x = [y] \quad \text{for some } \underline{y \in \mathbb{k}^{(V \times W)}}$$

$\Rightarrow$

$$y = \sum_{i=1}^{n} a_i \cdot (\vec{v}_i, \vec{w}_i)$$

for some $(\vec{v}_i, \vec{w}_i) \in V \times W$, $a_i \in k$

$$\Rightarrow \quad x = [y] = \left[\sum_{i=1}^{n} a_i \cdot (\vec{v}_i, \vec{w}_i) \right]$$

$$= \sum_{i=1}^{n} a_i \cdot \underbrace{[(\vec{v}_i, \vec{w}_i)]}_{} = \vec{v}_i \otimes \vec{w}_i$$

$$= \sum_{i=1}^{n} a_i \cdot \vec{v}_i \otimes \vec{w}_i$$

Remark

$$V \otimes W = k^{(V \times W)} \Big/ \operatorname{span} \left\{ \begin{array}{l} (v_1+v_2, w) - (v_1, w) - (v_2, w) \\ (v, w_1+w_2) - (v, w_1) - (v, w_2) \\ (cv, w) - c(v, w) \\ (v, cw) - c(v, w) \end{array} \right\}$$

$R = \operatorname{span}\{\ldots\}$

$$\Rightarrow \quad \overset{(v_1+v_2)\otimes w}{[(v_1+v_2, w)]} - \overset{v_1 \otimes w}{[(v_1, w)]} - \overset{v_2 \otimes w}{[(v_2, w)]} \in R$$

$$= [(v_1+v_2, w) - (v_1, w) - (v_2, w)]$$

$$= 0$$

$$\Rightarrow \quad (v_1+v_2)\otimes w - v_1\otimes w - v_2 \otimes w = 0$$

i.e. ① $(v_1+v_2)\otimes w = v_1\otimes w + v_2 \otimes w$

Similarly,

② $v\otimes(w_1+w_2) = v\otimes w_1 + v\otimes w_2$

→ ③ $(cv)\otimes w = c\cdot(v\otimes w)$

$$\begin{array}{ccc} (cv)\otimes w & = & c\cdot(v\otimes w) \\ \| & & \| \\ [(cv, w)] & & c\cdot[(v, w)] \end{array}$$

→ ④ $v\otimes(cw) = c\cdot(v\otimes w)$

Recall an element $x \in V\otimes W$ is of the form

$$x = \sum_{i=1}^{n} a_i \cdot (\vec{v}_i \otimes \vec{w}_i)$$

$$\overset{③}{=} \sum_{i=1}^{n} (\underbrace{a_i\cdot\vec{v}_i}_{\tilde{v}_i}) \otimes \vec{w}_i$$

$$= \sum_{i=1} v_i \otimes w_i$$

e.g. $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$

$$(1,2) \otimes (3,4)$$

$$= \big((1,0) + 2(0,1) \big) \otimes (3,4)$$

$$= (1,0) \otimes (3,4) + 2 \cdot (0,1) \otimes (3,4)$$

$$= (1,0) \otimes \big(3(1,0) + 4(0,1) \big)$$
$$+ 2 \cdot (0,1) \otimes \big(3(1,0) + 4(0,1) \big)$$

$$= 3 \cdot (1,0) \otimes (1,0) + 4 (1,0) \otimes (0,1)$$
$$+ 6 \cdot (0,1) \otimes (1,0) + \underbrace{8 (0,1) \otimes (0,1)}$$

$$\|$$

$$(0,8) \otimes (0,1)$$
$$= (0,1) \otimes (0,8)$$
$$= (0,4) \otimes (0,2)$$

Remark

The idea of defining $V \otimes W$:

$$\text{free object} \Big/ \text{relations}$$

e.g. $$k^{(\mathbb{R})} \Big/ \operatorname{span}\{1+1-3\}$$ ← "$1+1=3$" here

exer

A simple tensor $\vec{v} \otimes \vec{w} = 0$ in $V \otimes W$

$\Leftrightarrow \vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$

Note
$$\vec{0} \otimes \vec{w} = (0 \cdot \vec{0}) \otimes \vec{w}$$
$$= 0 \cdot (\vec{0} \otimes \vec{w}) = \vec{0} \text{ in } V \otimes W$$

Example

$$\dim(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}) = 1$$

<u>pf</u>

Any element $x \in \mathbb{R} \otimes \mathbb{R}$ is of the form

$$x = \sum_{i=1}^{n} \underbrace{a_i}_{= a_i \cdot 1} \otimes b_i$$

where $a_i, b_i \in \mathbb{R}$

$$x = \sum_{i=1}^{n} (a_i \cdot 1) \otimes b_i$$

$$\overset{(3)}{=} \sum_{i=1}^{n} a_i \cdot (1 \otimes \underbrace{b_i}_{= b_i \cdot 1})$$

$$\overset{(4)}{=} \underbrace{\sum_{i=1}^{n} a_i b_i}_{\in \mathbb{R}} \cdot (1 \otimes 1)$$

So $\mathbb{R} \otimes \mathbb{R} = \operatorname{span}\{ \underbrace{1 \otimes 1}_{\neq 0 \text{ in } \mathbb{R} \otimes \mathbb{R}} \}$

So $\{1 \otimes 1\}$ is a basis of $\mathbb{R} \otimes \mathbb{R}$

$\{1\otimes 1\}$ is a basis for $\mathbb{R}\otimes\mathbb{R}$

$$\Rightarrow \quad \dim(\mathbb{R}\otimes\mathbb{R}) = 1 \qquad \#$$

<u>Example</u>

$$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}^2$$

<u>pf</u>

Any element $x \in \mathbb{R}^2 \otimes \mathbb{R}$ is of the form

$$x = \sum_{i=1}^{n} \vec{v}_i \otimes \overset{a_i \cdot 1}{\overset{\|}{a_i}}, \qquad \begin{array}{l} \vec{v}_i \in \mathbb{R}^2 \\ a_i \in \mathbb{R} \end{array}$$

$$= \sum_{i=1}^{n} a_i \cdot (\vec{v}_i \otimes 1)$$

$$= \sum_{i=1}^{n} (a_i \vec{v}_i) \otimes 1$$

$$= \left(\sum_{i=1}^{n} a_i \vec{v}_i\right) \otimes 1$$

(*) So any element in $\mathbb{R}^2 \otimes \mathbb{R}$

is of the form $\vec{y} \otimes 1$ for some $\vec{y} \in \mathbb{R}^2$

Define

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \otimes \mathbb{R}$$

$$\phi(\vec{y}) = \vec{y} \otimes 1$$

$\Rightarrow$ ϕ is linear because

$$\phi(b_1\vec{y}_1 + b_2\vec{y}_2) = (b_1\vec{y}_1 + b_2\vec{y}_2) \otimes 1$$

$$= b_1 \cdot (\vec{y}_1 \otimes 1) + b_2 \cdot (\vec{y}_2 \otimes 1)$$

$$= b_1 \cdot \phi(\vec{y}_1) + b_2 \cdot \phi(\vec{y}_2)$$

$\circledast$ $\Rightarrow$ ϕ is onto

ϕ is 1-1 because

$\phi(\vec{u}) = 0$

$$\underset{\vec{y}\otimes 1}{\underset{\parallel}{F(\vec{y})}} = 0 \Rightarrow y = 0$$

#

More generally, let $V_1, \cdots, V_n$ be vector spaces over $\mathbb{k}$.

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n = \frac{\mathbb{k}^{(V_1 \times V_2 \times \cdots \times V_n)}}{\operatorname{Span}\left\{ \begin{array}{l} ① \ (\vec{v}_1, \cdots, \vec{v}_i + \vec{v}_i', \cdots \vec{v}_n) - (\vec{v}_1, \cdots, \vec{v}_i, \cdots, \vec{v}_n) - (\vec{v}_1, \cdots, \vec{v}_i', \cdots, \vec{v}_n) \\ ② \ (\vec{v}_1, \cdots, c\cdot\vec{v}_i, \cdots, \vec{v}_n) - c\cdot(\vec{v}_1, \cdots, \vec{v}_i, \cdots, \vec{v}_n) \end{array} \right\}}$$

Notation:

$$[(\vec{v}_1, \cdots, \vec{v}_n)] = \vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_n$$

So an element x in $V_1 \otimes \cdots \otimes V_n$ is of the form

$$x = \sum_{j=1}^{m} \vec{v}_1^{\,j} \otimes \cdots \otimes \vec{v}_n^{\,j}$$

e.g.

$$x = \vec{v} \otimes \vec{w} \otimes \vec{z} + \vec{p} \otimes \vec{q} \otimes \vec{r}$$

The elements satisfy the relations

$$\vec{v}_1 \otimes \cdots \otimes (a\vec{v}_i + b\vec{v}_i\,') \otimes \cdots \otimes \vec{v}_n$$
$$= a \cdot (\vec{v}_1 \otimes \cdots \vec{v}_i \otimes \cdots \otimes \vec{v}_n)$$
$$+ b \cdot (\vec{v}_1 \otimes \cdots \vec{v}_i\,' \otimes \cdots \otimes \vec{v}_n)$$

§ Universal property

Universal property for (as free modules) vector spaces

Recall (from HW1)

Prop

Let V, W be ~~a~~ vector spaces and $\beta = \{v_\lambda\}_{\lambda \in V}$ be a basis for V.

Given any map

$$f : \Lambda \to W, \quad \lambda \mapsto \underset{f(\lambda)}{\overset{\shortparallel}{w_\lambda}} \in W$$

$\exists!$ linear map

$$T_f : V \longrightarrow W$$

s.t. $\quad T_f(v_\lambda) = w_\lambda \quad \forall \lambda \in \Lambda$

$$\begin{tikzcd}
\lambda \arrow[r, mapsto] & w_\lambda \\
\Lambda \arrow[r, "\forall f"] \arrow[d] & W \\
V \arrow[ur, dashed, "\exists!\ \text{linear}\ T_f"'] &
\end{tikzcd}$$

$\lambda \mapsto v_\lambda$ basis

<u>Thm</u> (Universal property for $\mathbb{k}^{(\beta)}$)

Let β be an arbitrary set, and $\mathbb{k}^{(\beta)}$ be the vector space freely generated by β

Let W be a vector space and $f : \beta \to W$ be a map of sets.

Then $\exists!$ linear map

$$T_f : \mathbb{k}^{(\beta)} \longrightarrow W$$

s.t. the diagram

$$\begin{tikzcd} B \arrow[r, "\text{map of sets}"'] \arrow[d, "\iota"] & W \\ \Bbbk^{(B)} \arrow[ur, dashed, "\exists!\, T_f"', "\text{linear}"] & \end{tikzcd}$$

commutes (i.e. $f = T_f \circ \iota$),

where $\iota : B \to \Bbbk^{(B)}$, $\underset{x}{\overset{B}{\in}} \mapsto 1 \cdot x$

Remark

This property implies

$$\left\{ B \xrightarrow[\text{map of sets}]{\text{arbitrary}} W \right\} \xleftrightarrow[\text{onto}]{\text{1-1}} \left\{ \Bbbk^{(B)} \xrightarrow[\text{maps}]{\text{linear}} W \right\}$$

Theorem 2.8. *Let $V_1, \cdots, V_k$ and W be finite-dimensional vector spaces. Let*

$$\beta_p = \{v_1^p, \cdots, v_{n_p}^p\} \quad \text{and} \quad \gamma = \{w_1, \cdots, w_m\}$$

be bases for V_p and W, respectively. A multilinear map

$$f : V_1 \times \cdots \times V_k \to W$$

is uniquely determined by the $n_1 \cdots n_k$ vectors

$$f(v_{i_1}^1, v_{i_2}^2, \cdots, v_{i_k}^k), \qquad 1 \le i_p \le n_p,$$

in W. Furthermore, assume $f_{i_1,\cdots,i_k}^j \in \Bbbk$ are the numbers such that

$$f(v_{i_1}^1, v_{i_2}^2, \cdots, v_{i_k}^k) = \sum_{j=1}^{m} f_{i_1,\cdots,i_k}^j w_j. \tag{5}$$

The map

$$\begin{aligned} \Big\{\text{multilinear } V_1 \times \cdots \times V_k \to W\Big\} &\longrightarrow \left\{ \left(a_{i_1,\cdots,i_k}^j\right)_{\substack{1\le j\le m \\ 1\le i_p \le n_p}} \,\middle|\, a_{i_1,\cdots,i_k}^j \in \Bbbk \right\}, \\ f \qquad &\longmapsto \qquad \left(f_{i_1,\cdots,i_k}^j\right)_{\substack{1\le j\le m \\ 1\le i_p \le n_p}}, \end{aligned}$$

is an isomorphism of vector spaces.

The collection of numbers $\left(f_{i_1,\cdots,i_k}^j\right)_{\substack{1\le j\le m \\ 1\le i_p \le n_p}}$ is sometimes referred as a **coordinate representation** of the k-linear map f.

Definition [edit]

Given a set of basis vectors $\{\mathbf{e}_i\}$ for the underlying vector space of the algebra, the **structure constants** or **structure coefficients** $c_{ij}{}^k$ express the multiplication $\cdot$ of pairs of vectors as a linear combination:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \sum_k c_{ij}{}^k \mathbf{e}_k.$$

bilinear

The upper and lower indices are frequently not distinguished, unless the algebra is endowed with some other structure that would require this (for example, a pseudo-Riemannian metric, on the algebra of the indefinite orthogonal group so(p,q)). That is, structure constants are often written with all-upper, or all-lower indexes. The distinction between upper and lower is then a convention, reminding the reader that lower indices behave like the components of a dual vector, i.e. are covariant under a change of basis, while upper indices are contravariant.

The structure constants obviously depend on the chosen basis. For Lie algebras, one frequently used convention for the basis is in terms of the ladder operators defined by the Cartan subalgebra; this is presented further down in the article, after some preliminary examples.

Example: Lie algebras [edit]

For a Lie algebra, the basis vectors are termed the generators of the algebra, and the product is given by the Lie bracket. That is, the algebra product $\cdot$ is *defined* to be the Lie bracket: for two vectors A and B in the algebra, the product is $A \cdot B \equiv [A, B]$. In particular, the algebra product $\cdot$ *must not* be confused with a matrix product, and thus sometimes requires an alternate notation.

There is no particular need to distinguish the upper and lower indices in this case; they can be written all up or all down. In physics, it is common to use the notation T_i for the generators, and $f_{ab}{}^c$ or f^{abc} (ignoring the upper-lower distinction) for the structure constants. The Lie bracket of pairs of generators is a linear combination of generators from the set, i.e.

$$[T_a, T_b] = \sum_c \cancel{f_{ab}{}^{c}}\, T_c.$$

$[e_i, e_j] = \sum_k C_{ij}^k e_k$

structure constants

Christoffel symbols

From Wikipedia, the free encyclopedia

total diff $\rightsquigarrow \nabla : V \times W \to W$ $\quad \nabla_{e_i} \tilde{e}_j = \Gamma_{ij}^k \tilde{e}_k$

In mathematics and physics, the **Christoffel symbols** are an array of numbers describing a metric connection.[1] The metric connection is a specialization of the affine connection to surfaces or other manifolds endowed with a metric, allowing distances to be measured on that surface. In differential geometry, an affine connection can be defined without reference to a metric, and many additional concepts follow: parallel transport, covariant derivatives, geodesics, etc. also do not require the concept of a metric.[2][3] However, when a metric is available, these concepts can be directly tied to the "shape" of the manifold itself; that shape is determined by how the tangent space is attached to the cotangent space by the metric tensor.[4] Abstractly, one would say that the manifold has an associated (orthonormal) frame bundle, with each "frame" being a possible choice of a coordinate frame. An invariant metric implies that the structure group of the frame bundle is the orthogonal group $\mathrm{O}(p, q)$. As a result, such a manifold is necessarily a (pseudo-)Riemannian manifold.[5][6] The Christoffel symbols provide a concrete representation of the connection of (pseudo-)Riemannian geometry in terms of coordinates on the manifold. Additional concepts, such as parallel transport, geodesics, etc. can then be expressed in terms of Christoffel symbols.

In general, there are an infinite number of metric connections for a given metric tensor; however, there is a unique connection that is free of torsion, the Levi-Civita connection. It is common in physics and general relativity to work almost exclusively with the Levi-Civita connection, by working in coordinate frames (called holonomic coordinates) where the torsion vanishes. For example, in Euclidean spaces, the Christoffel symbols describe how the local coordinate bases change from point to point.

At each point of the underlying n-dimensional manifold, for any local coordinate system around that point, the Christoffel symbols are denoted $\Gamma^i{}_{jk}$ for $i, j, k = 1, 2, \ldots, n$. Each entry of this $n \times n \times n$ array is a real number. Under *linear* coordinate transformations on the manifold, the Christoffel symbols transform like the components of a tensor, but under general coordinate transformations (diffeomorphisms) they do not. Most of the algebraic properties of the Christoffel symbols follow from their relationship to the affine connection; only a few follow from the fact that the structure group is the orthogonal group $\mathrm{O}(m, n)$ (or the Lorentz group $\mathrm{O}(3, 1)$ for general relativity).

Christoffel symbols are used for performing practical calculations. For example, the Riemann curvature tensor can be expressed entirely in terms of the Christoffel symbols and their first partial derivatives. In general relativity, the connection plays the role of the gravitational force field with the corresponding gravitational potential being the metric tensor. When the coordinate system and the metric tensor share some symmetry, many of the $\Gamma^i{}_{jk}$ are zero.

The Christoffel symbols are named for Elwin Bruno Christoffel (1829–1900).[7]

10. Let $\mathbb{N}$ be the set of positive integers, and $V_n = \mathbb{R}$ for any $n \in \mathbb{N}$. Show that a basis for $\bigoplus_{n\in\mathbb{N}} V_n = \bigoplus_{n\in\mathbb{N}} \mathbb{R}$ is countable, but a basis for $\prod_{n\in\mathbb{N}} V_n = \prod_{n\in\mathbb{N}} \mathbb{R}$ is uncountable.

Solution: Nontrivial part: show that the set $\{(1, t^1, t^2, \cdots) \mid t \in \mathbb{R}\}$ is linearly independent. Key:

i.e. $a_1(1, t_1^1, t_1^2, \cdots) + a_2(1, t_2^1, t_2^2, \cdots) + \cdots + a_n(1, t_n^1, t_n^2, \cdots) = 0$ (*)

$$\det\begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & \cdots & t_n^{n-1} \end{pmatrix} = \det\begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 0 & t_2 - t_1 & \cdots & t_2^{n-1} - t_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & t_n - t_1 & \cdots & t_n^{n-1} - t_1^{n-1} \end{pmatrix}$$

$$= (t_2 - t_1)\cdots(t_n - t_1)\det\begin{pmatrix} 1 & \cdots & \sum_{i=0}^{n-2} t_1^i t_2^{n-2-i} & \sum_{i=0}^{n-1} t_1^i t_2^{n-1-i} \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & \sum_{i=0}^{n-1} t_1^i t_{n-1}^{n-2-i} & \sum_{i=0}^{n-1} t_1^i t_{n-1}^{n-1-i} \end{pmatrix}$$

$$= (t_2 - t_1)\cdots(t_n - t_1)\det\begin{pmatrix} 1 & \cdots & \sum_{i=0}^{n-2} t_1^i t_2^{n-2-i} & t_2^{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & \sum_{i=0}^{n-1} t_1^i t_{n-1}^{n-2-i} & t_{n-1}^{n-1} \end{pmatrix}$$

$$= \cdots = (t_2 - t_1)\cdots(t_n - t_1)\det\begin{pmatrix} 1 & \cdots & t_2^{n-2} & t_2^{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & t_{n-1}^{n-2} & t_{n-1}^{n-1} \end{pmatrix}$$

$$= \cdots = \prod_{i<j}(t_j - t_i).$$

(*) $\Rightarrow$ $a_1(1, \cdots, t_1^{n-1}) + \cdots + a_n(1, \cdots, t_n^{n-1}) = 0$

$\Rightarrow a_1 = \cdots = a_n = 0$ when $t_1, \cdots, t_n$ are distinct

11. Show that there exist vector spaces W and V_α, $\alpha \in I$, such that the linear map

$$\Psi : \bigoplus_{\alpha\in I} \operatorname{Hom}(W, V_\alpha) \to \operatorname{Hom}\Big(W, \bigoplus_{\alpha\in I} V_\alpha\Big), \quad \Psi\Big(\sum_{\alpha\in I} T_\alpha\Big)(\vec{w}) = \sum_{\alpha\in I} T_\alpha(\vec{w})$$

is not onto.