Linear Algebra 10/5

Quotient space

Let V be a vector space.

W be a vector subspace of V.

Define (for $v_1, v_2 \in V$)

$$v_1 \sim v_2 \iff v_1 - v_2 \in W$$

Lemma

$\sim$ is an equivalence relation

Def

The quotient space V/W is the set

$V/\sim$ of equivalence classes equipped

with the operation

$$a_1 \underbrace{[v_1]}_{= \{\vec{v} \in V \mid \vec{v} \sim v_1\}} + a_2[v_2] = [a_1 v_1 + a_2 v]$$

for any $a_1, a_2 \in \mathbb{k}$, $v_1, v_2 \in V$.

Remark (well-def. problem)

You may have

$$\underbrace{[v_1] = [\tilde{v}_1]}_{\iff v_1 - v_2 \in W} \quad \text{but} \quad v_1 \neq \tilde{v}_1$$

Q: $[v_1] = [\tilde{v}_1]$, $[v_2] = [\tilde{v}_2]$

$\overset{?}{\Rightarrow}$ $[a_1 v_1 + a_2 v_2] = [a_1 \tilde{v}_1 + a_2 \tilde{v}_2]$

Ans is 'Yes' (You should check)

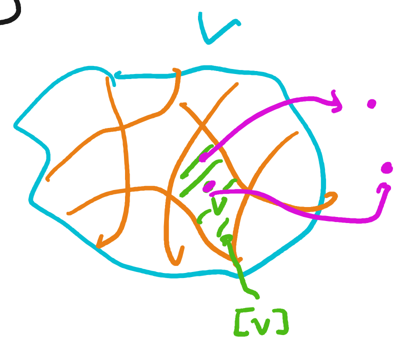
Prop

V/W is a vector space which is accompanied by a natural projection map

$$\pi : V \longrightarrow V/W$$

linear

$$\pi(v) = [v]$$



Furthermore, suppose $\tilde{V}$ is another vector space. and $T : V \to \tilde{V}$ is linear.

Then $\exists$ linear map

$$\tilde{T} : V/W \longrightarrow \tilde{V}$$

s.t. $T = \tilde{T} \circ \pi$

$\Longleftrightarrow$ $T(W) = \{0\}$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & \tilde{V} \\ V/W \arrow[ur, dashed, "\tilde{T}"'] & \end{tikzcd}$$

$v \mapsto [v]$

hope: $\tilde{T}([v]) = T(v)$

Example

Let $V = \mathbb{R}^2$, $W = \left\{ \binom{x_1}{x_2} \in \mathbb{R}^2 \;\middle|\; x_1 + x_2 = 0 \right\} = \mathrm{Span}\left\{ \binom{1}{-1} \right\} \cong \mathbb{R}$

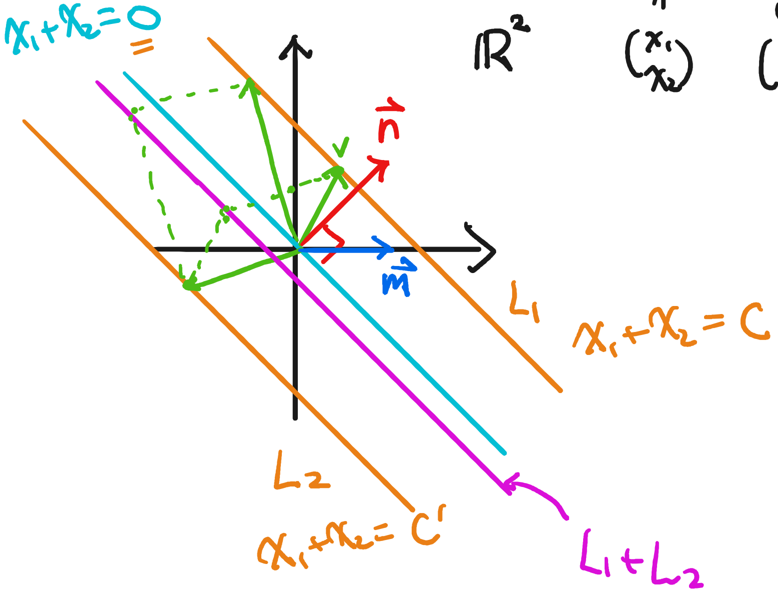
$V/W = \{ [\vec{v}] \mid \vec{v} \in V \} = \{ \ \leftarrow x_1 + x_2 = c \ \}$

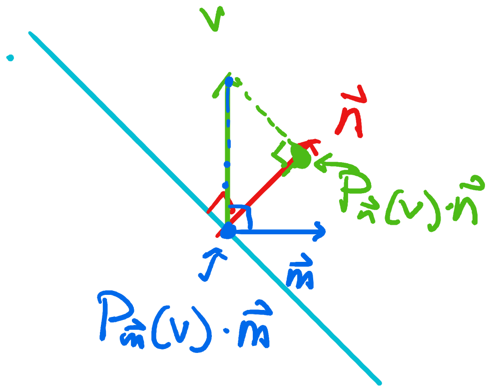
$/W = \{[v] \mid v \in V\}$ $\qquad c \in \mathbb{R}$

$$[v_1] = [v_2] \iff \underbrace{v_1 - v_2}_{=\binom{x_1-y_1}{x_2-y_2}} \in W \iff (x_1-y_1)+(x_2-y_2)=0$$

$$v_1 = \binom{x_1}{x_2}, \quad v_2 = \binom{y_1}{y_2}$$

$$\iff \underline{x_1+x_2} = y_1+y_2 = c$$





Let $\vec{n} = \binom{1}{1}$, $\vec{m} = \binom{1}{0}$.

$$P_{\vec{n}}(v) = \frac{\langle v, \vec{n}\rangle}{\langle \vec{n}, \vec{n}\rangle}, \qquad P_{\vec{m}}(v) = \frac{\langle v, \vec{m}\rangle}{\langle \vec{m}, \vec{m}\rangle}$$

If we define

$$\tilde{P}_{\vec{n}}([v]) = P_{\vec{n}}(v)$$

$$\rightarrow \tilde{P}_{\vec{m}}([v]) = P_{\vec{m}}([v]) \quad \times$$

then

① $\tilde{P}_{\vec{n}}$ is well-defined because $P_{\vec{n}}(W)=0$

$$P_{\vec{n}}\binom{1}{-1} = \frac{1\cdot 1 + (-1)\cdot 1}{\langle \vec{n}, \vec{n}\rangle} = 0$$

② $\tilde{P}_{\vec{m}}$ is NOT well-defined

because $\qquad (\vec{m} = \binom{1}{0})$

$$\tilde{P}_{\vec{m}}\left(\left[\binom{1}{-1}\right]\right) = \frac{\langle \binom{1}{-1}, \binom{1}{0}\rangle}{\cdots} = \frac{1}{\cdots} = 1$$

$\langle m, m \rangle$

$$\tilde{P}_{\vec{m}}\left(\left[\begin{pmatrix}0\\0\end{pmatrix}\right]\right) = O$$

but

$$\left[\begin{pmatrix}1\\-1\end{pmatrix}\right] = \left[\begin{pmatrix}0\\0\end{pmatrix}\right] =$$

$x_1 + x_2 = 0$

$\#$

<u>Thm</u> (Isomorphism thm)

Let $T : V \to W$ be a linear map

Then

$$V/\ker(T) \cong \operatorname{im}(T)$$

Note: $\dim(V/W) = \dim V - \dim W$

Let A be an $m \times n$ matrix, and $T_A : k^n \to k^m$

$$T_A(x) = Ax$$

Define

$$\operatorname{rank}(A) := \dim(\operatorname{im}(T_A))$$

$$\operatorname{nullity}(A) := \dim(\ker(T_A))$$

$=$ #(free variables) in the sol of

$$A\vec{x} = \vec{0}$$

<u>Cor</u>

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

§2 Tensor product

Motivation:

① Consider

$$\mu : \mathbb{R}\times\mathbb{R} \to \mathbb{R},$$

$$\mu(a,b) = a\cdot b$$ multiplication

As a map

$$\mathbb{R}\oplus\mathbb{R} \to \mathbb{R}$$

is NOT linear, because

$$\mu(c\cdot(a,b)) = \mu(c\cdot a,\ c\cdot b)$$

$$= (c\cdot a)\cdot(c\cdot b) = c^2\cdot(a\cdot b)$$

$$= c^2\mu(a,b) \neq c\,\mu(a,b)$$

To study μ by the theory of linear maps, we will introduce a vector sp.

$$\mathbb{R}\otimes\mathbb{R}$$

The map

$$\mathbb{R}\otimes\mathbb{R}\to\mathbb{R},\quad a\otimes b\longmapsto \mu(a,b)$$

is a linear map !!

② Recall

$$\left\{\begin{array}{c}\text{iso class}\\ \text{of finite dim'l}\\ \text{vector spaces}\end{array}\right\} \xleftrightarrow[\text{1-1, onto}]{\dim} \mathbb{Z}_{\geq 0}$$

$$\oplus \longleftrightarrow +$$

$$??\ \otimes \longleftrightarrow \times$$

§2.1 Multilinear map

Def

Let $V_1,\cdots,V_k$, W be vector spaces

A map

$$f: V_1\times\cdots\times V_k \longrightarrow W$$

is called a k-linear (or multilinear) map if

$$f(v_1 \cdots, v_{i-1}, av_i+bv_i', v_{i+1},\cdots, v_k)$$

$$= a f(v_1,\cdots, v_i, \cdots, v_k)$$

$+ b\, f(v_1, \ldots, v_i', \ldots v_k)$

$\forall\ i = 1, \cdots, k,\ a, b \in \mathbb{k},\ v_i, v_i' \in V_i.$

<u>Example</u>

① A 2-linear map is also called a bilinear map

The map

$$\mu : \mathbb{k} \times \mathbb{k} \to \mathbb{k},\quad \mu(a,b) = ab$$

is a bilinear map, because

$$\mu(c_1 a_1 + c_2 a_2, b) \quad \text{fix}$$

$$= (c_1 a_1 + c_2 a_2) b = c_1 \overbrace{a_1 b}^{\mu(a_1,b)} + c_2 \overbrace{a_2 b}^{\mu(a_2,b)}$$

$$= c_1 \mu(a_1, b) + c_2 \mu(a_2, b)$$

i.e. fix b $x \mapsto \mu(x,b)$ is linear

and similarly

$$\mu(a, c_1 b_1 + c_2 b_2)$$

$$= c_1 \mu(a, b_1) + c_2 \mu(a, b_2)$$

More generally, the map

More generally, the map

$$\mu^{\ell} : \overbrace{\Bbbk \times \cdots \times \Bbbk}^{\ell+1} \longrightarrow \Bbbk$$

$$\mu^{\ell}(a_0, \dots, a_\ell) = a_0 \cdot \cdots \cdot a_\ell$$

is an $(\ell+1)$-linear map.

② Any inner product

$$\langle\, ,\, \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is a bilinear map, becauce

$$\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$$

$$\langle x, ay+bz\rangle = a\langle x,y\rangle + b\langle x,z\rangle$$

③ The pairing

$$\langle \mid \rangle : V \times V^{\vee} \longrightarrow \Bbbk, \quad \langle \vec{v}, \xi\rangle = \xi(\vec{v})$$

is bilinear

④ Let $H : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ be

$$H\left(\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right)$$

$$\Rightarrow z_1 w_1 + \cdots + z_n w_n$$

for $z_i, w_j \in \mathbb{C}$.

H is bilinear over $\mathbb{R}$ (think $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector sp's)

but it's NOT bilinear over $\mathbb{C}$

$$\left(H(\vec{z}, c\vec{w}) = \bar{c}\, H(\vec{z}, \vec{w}), \text{ for } c \in \mathbb{C} \right)$$

⑤ The matrix multiplication

$$M_{m\times p}(\Bbbk) \times M_{p\times n}(\Bbbk) \longrightarrow M_{m\times n}(\Bbbk)$$
$$(A, B) \longmapsto AB$$

is bilinear ($\because$ $(c_1A_1 + c_2A_2)B = c_1(A_1B) + c_2(A_2B)$

and $A(c_1B_1 + c_2B_2) = c_1AB_1 + c_2AB_2$)

The induced "Lie bracket"

$$M_n(\Bbbk) \times M_n(\Bbbk) \to M_n(\Bbbk)$$
$$(A, B) \longmapsto [A, B] := AB - BA$$

is bilinear.

⑥ The map

$$\mathbb{k}^n \times \cdots \times \mathbb{k}^n \longrightarrow \mathbb{k}$$

$$\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right) \longmapsto \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is an n-linear map

Recall: $\det \begin{pmatrix} c \cdot a_{11} & a_{12} & \cdots \\ \vdots & \vdots & \\ c \cdot a_{n1} & a_{n2} & \end{pmatrix} = c \det \begin{pmatrix} \cdots \\ \vdots \ \ddots \ \vdots \\ \cdots \end{pmatrix}$

$\Rightarrow \quad \det(c \cdot A) = c^n \det A \neq c \cdot \det A$

<u>Prop</u>

The space of k-linear maps

$$V_1 \times \cdots \times V_k \longrightarrow W$$

form a vector sp.

$$(a_1 f_1 + a_2 f_2)(v_1, \cdots, v_k)$$

in W

$$= a_1 \cdot f_1(v_1, \cdots, v_k) + a_2 \cdot f_2(v_1, \cdots, v_k)$$

pf: exer.

<u>Remark</u>

linear map = 1-linear map

matrix representation

k-linear map $V_1 \times \cdots \times V_k \longrightarrow W$ ⟷ something like matrix represent ??

$$\left(a^{j}_{i_1, \dots, i_k} \right)_{\substack{1 \le j \le m \\ 1 \le i_p \le n_p}}$$

"Coordinate representation"

$m = \dim W$, $n_p = \dim V_p$

- a collection $\left(a^{j}_{i_1, \dots, i_k} \right)$ has $n_1 \cdot n_2 \cdots n_k \cdot m$ numbers
- a collection is order-sensitive
- all these collections of "the same type" form a vector space withe the componentwise operations

Thm

Let $V_1, \dots, V_k$ and W be finite-dim'l vector spaces. Let

$\beta_p = \{v_1^p, \dots, v_{n_p}^p\}$ basis for V_p

$\gamma = \{w_1, \dots, w_m\}$ basis for W.

A multilinear map

$$f: V_1 \times \cdots \times V_k \longrightarrow W$$

is uniquely determined by

$$f(v_{i_1}^1, v_{i_2}^2, \dots, v_{i_k}^k) \in W, \quad 1 \le i_p \le n_p$$

$n_1 \cdot n_2 \cdots n_k$ vectors

$\because \gamma$ is a basis

$\therefore \exists!\ f_{i_1, \dots, i_k}^{j} \in \mathbb{k}$ s.t.

$$= \sum_{j=1}^{m} f_{i_1, \dots, i_k}^{j} w_j$$

The map

$$\left\{ \begin{array}{c} \text{multilinear} \\ V_1 \times \cdots \times V_k \to W \end{array} \right\} \longrightarrow \left\{ \left(a_{i_1, \dots, i_k}^{j}\right)_{\substack{1 \le j \le m \\ 1 \le i_p \le n_p}} \;\middle|\; a_{i_1, \dots, i_k}^{j} \in \mathbb{k} \right\}$$

$$f \longmapsto \left(f_{i_1, \dots, i_k}^{j}\right)_{\substack{1 \le j \le m \\ 1 \le i_p \le n_p}}$$

is an iso of vector spaces.

pf: exer.

<u>Example</u>

Let $f: \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ be the map

$$f(z_1, z_2) = z_1 \cdot \overline{z_2}$$

Let $\beta = \{ 1, i \}$ — a basis for $\mathbb{C}$ $\cong \mathbb{R}^2$ over $\mathbb{R}$

$\quad\; \begin{matrix} \shortparallel & \shortparallel \\ v_1 & v_2 \end{matrix}$

$\Rightarrow$ f is bilinear

$$f(1,1) = 1 \cdot \bar{1} = \underset{f_{11}^1}{1} \cdot 1 + \underset{f_{11}^2}{0} \cdot i$$

$$f(1, i) = 1 \cdot \bar{i} = 0 \cdot 1 + (-1) \cdot i$$

$$f(i, 1) = i \cdot \bar{1} = 0 \cdot 1 + 1 \cdot i$$

$$f(i, i) = i \cdot \bar{i} = 1 \cdot 1 + 0 \cdot i$$

$\Rightarrow$ $f_{11}^1 = 1$, $f_{12}^1 = 0$, $f_{21}^1 = 0$, $f_{22}^1 = 1$

$f_{11}^2 = 0$, $f_{12}^2 = -1$, $f_{21}^2 = 1$, $f_{22}^2 = 0$

#

Einstein summation convention

e.g. $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}$

Write $\displaystyle\sum_{i=1}^{3} c_i \vec{e}_i = c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$

$= c^i \vec{e}_i \leftarrow$ $\displaystyle\sum_{i=1}^{3} c^i \vec{e}_i$ 的缩写

same index $\Rightarrow$ sum over i

e.g.

$$f(v^1_{i_1}, v^2_{i_2}, \cdots, v^k_{i_k}) = \underline{f^{j}_{i_1,\cdots,i_k} w_j}$$

↑ actually means

$$\sum_{j=1}^{m} f^{j}_{i_1,\cdots,i_k} w_j$$

<u>Example</u>

① Let

$\beta = \{v_1, \cdots, v_n\}$ basis for V

$\gamma = \{w_1, \cdots, w_m\}$ basis for W

$T: V \to W$ be linear

Let a^i_j be the numbers satisfying

$$T(v_j) = a^1_j w_1 + \cdots + a^m_j w_m$$

$$= \underline{a^i_j w_i} \qquad \forall j = 1, \cdots, n$$

i.e. (a^i_j) = matrix repn of T

Let $T': W \to W'$ be another linear map

with matrix repn $((b^j_{\;}))$

with matrix repn (b_i^l)

$\Rightarrow$

$$(T' \circ T)(v_j) = T'(a_j^i w_i)$$

$$= a_j^i T'(w_i)$$

$$= a_j^i (b_i^l w'_l)$$

actually means $\sum_{i=1}^{m} \sum_{l=1}^{m'} a_j^i b_i^l \vec{w'_l}$

$$= \boxed{(a_j^i b_i^l)} w'_l$$

$$\| $$

$$\sum_{i=1}^{m} a_j^i b_i^l = (j,l)\text{-entry of matrix multiplication}$$

② Let $f: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $f(z_1, z_2) = z_1 \cdot \bar{z}_2$.

A vector in $\mathbb{C}$ is of the form

$$a^j v_j = a^1 v_1 + a^2 v_2 = a^1 + a^2 i$$

$\Rightarrow$

$\because$ bilinear

$$f(a^j v_j, b^k v_k) = a^j f(v_j, b^k v_k)$$

$$= a^j b^k f(v_j, v_k)$$

$$= a^j b^k f_{jk}^l v_l$$

which means

$$f\left(\sum_{j=1}^{2} a^j v_j, \sum_{k=1}^{2} b^k v_k\right) = \sum_{j,k,l=1} a^j b^k f_{jk}^{l} v_l$$

in the usual notation.

$$f(a^1 + a^2 i, \; b^1 + b^2 i)$$

$$= a^1 f(1, b^1 + b^2 i) + a^2 f(i, b^1 + b^2 i)$$

$$= a^1 b^1 f(1, 1) + a^1 b^2 f(1, i)$$

$$+ a^2 b^1 f(i, 1) + a^2 b^2 f(i, i)$$

$$= a^1 b^1 (f_{11}^1 \cdot 1 + f_{11}^2 \cdot i) + a^1 b^2 (f_{12}^1 \cdot 1 + f_{12}^2 \cdot i)$$

$$+ a^2 b^1 (f_{21}^1 \cdot 1 + f_{21}^2 \cdot i) + a^2 b^2 (f_{22}^1 \cdot 1 + f_{22}^2 \cdot i)$$