Linear Algebra 9/21

Constructions of vector spaces

Vector space freely generated by a set B

$= \mathbb{k}^{(B)}$ ← our notation

A simple case: $B = \{ \odot, \otimes \}$

$$\mathbb{k}^{(B)} := \{ a\odot + b\otimes \mid a, b \in \mathbb{k} \}$$

with the following rules

(i) $a\odot + b\otimes = a'\odot + b'\otimes$

$\iff a = a'$ and $b = b'$

(ii) $c(a\odot + b\otimes) + d(a'\odot + b'\otimes)$

$$= (ca + da')\odot + (cb + db')\otimes$$

Note that the map

$$\mathbb{k}^2 \to \mathbb{k}^{(B)} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto a\odot + b\otimes$$

is an iso of vector spaces.

<u>General case</u>: B is an arbitrary set

$\mathbb{k}^{(B)}$: Call the maps $B \to \mathbb{k}$ [illegible] s.t. [illegible]

$\mathbb{k}^{(B)} := \{$ all the maps $B \to \mathbb{k}$, $x \mapsto a_x$, s.t. $a_x = 0$ except for finitely many elements $x \in B$ $\}$

common notation "formal sum"

$$= \left\{ \sum a_x x \;\middle|\; a_x = 0 \text{ except finite } x \in B \right\}$$

with the rules

(i) $\sum a_x x = \sum b_x x \iff a_x = b_x \quad \forall x \in B$

(ii) $c(\sum a_x x) + d(\sum b_x x)$

$$= \sum (c a_x + d b_x)\, x$$

The map

$$B \to \mathbb{k}^{(B)}, \quad x \mapsto 1_{\mathbb{k}} \cdot x \in \mathbb{k}^{(B)}$$

is one-to-one. So B can be viewed a subset of $\mathbb{k}^{(B)}$.

<u>Prop</u>

The space $\mathbb{k}^{(B)}$ is a vector space, and B is a basis for $\mathbb{k}^{(B)}$.

<u>pf</u>

B is linearly indep: $\forall x_1, \ldots, x_n \in B$ with

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \underset{\sim}{0}$$ ← i.e. $a_x = 0$ $\forall x \in \beta$

$$\overset{(i)}{\Rightarrow} \quad a_1 = a_2 = \cdots = a_n = 0$$

$\text{span } \beta = \mathbb{k}^{(\beta)}$:

Any element in $\mathbb{k}^{(\beta)}$ is of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

for some $x_1, \cdots, x_n \in \beta$, which is in span β #

Direct sum and direct product

Let V and W be vector spaces.

The direct sum $V \oplus W$ is

$$V \oplus W = \{ (\vec{v}, \vec{w}) \mid \vec{v} \in V, \vec{w} \in W \}$$

with the rules

(i) $(\vec{v}_1, \vec{w}_1) = (\vec{v}_1, \vec{w}_2) \Leftrightarrow \vec{v}_1 = \vec{v}_2$ and $\vec{w}_1 = \vec{w}_2$

(ii) $a_1(\vec{v}_1, \vec{w}_1) + a_2(\vec{v}_2, \vec{w}_2) = (a_1\vec{v}_1 + a_2\vec{v}_2, a_1\vec{w}_1 + a_2\vec{w}_2)$

Let V_α, $\alpha \in \underset{\sim}{I}$, be vector spaces ← some index set, maybe $|I| = \infty$

There are 2 different generalizations of $V \oplus W$. ← "direct sum" and "direct product"

Def

The direct sum is

$$\bigoplus_{\alpha\in I} V_\alpha := \left\{ \begin{array}{l} \text{all maps } I \to \coprod_{\alpha\in I} V_\alpha,\ \alpha \mapsto \vec{v}_\alpha, \\ \text{s.t. } \vec{v}_\alpha \in V_\alpha\ \forall \alpha\in I, \text{ and} \\ \qquad \vec{v}_\alpha = \vec{0}_\alpha \text{ except finit } \alpha\in I \end{array} \right\}$$

$$= \left\{ \sum \vec{v}_\alpha \mid \vec{v}_\alpha \in V_\alpha,\ \vec{v}_\alpha = 0 \text{ except finite } \alpha\in I \right\}$$

(finite formal sum)

The direct product is

$$\prod_{\alpha\in I} V_\alpha := \left\{ \begin{array}{l} \text{all maps } I \to \coprod_{\alpha\in I} V_\alpha,\ \alpha \mapsto \vec{v}_\alpha, \text{ s.t.} \\ \qquad \vec{v}_\alpha \in V_\alpha \quad \forall \alpha\in I \end{array} \right\}$$

$$= \left\{ \sum \vec{v}_\alpha \mid \vec{v}_\alpha \in V_\alpha \right\}$$ (arbitrary formal sum)

In both $\bigoplus_{\alpha\in I} V_\alpha$ and $\prod_{\alpha\in I} V_\alpha$, we have

(i) $\sum \vec{v}_\alpha = \sum \vec{w}_\alpha \iff \vec{v}_\alpha = \vec{w}_\alpha$ (in V_α) $\forall \alpha\in I$

(ii) $a\left(\sum \vec{v}_\alpha\right) + b\left(\sum \vec{w}_\alpha\right) = \sum \left(a\vec{v}_\alpha + b\vec{w}_\alpha\right)$

Remark

If $|I| < \infty$, then $\bigoplus_{\alpha\in I} V_\alpha = \prod_{\alpha\in I} V_\alpha$.

Remark (HW 2)

① If $\mathcal{B}_\alpha$ is a basis for V_α for each

$\alpha \in I$, then $\coprod_{\alpha\in I} \beta_\alpha$ is a basis for $\bigoplus_{\alpha \in I} V_\alpha$ $\left(\Rightarrow \dim\left(\bigoplus_{\alpha\in I} V_\alpha\right) = \sum_{\alpha \in I} \dim V_\alpha \right)$

② If $V_\alpha = \mathbb{k}$ $\forall \alpha \in I$, then

$$\bigoplus_{\alpha \in I} V_\alpha = \mathbb{k}^{(I)}$$

③ A basis for $\bigoplus_{n\in\mathbb{N}} \overset{\text{"}V_n\text{"}}{\mathbb{R}}$ is countable

" $\prod_{n\in\mathbb{N}} \mathbb{R}$ is uncountable

Hom space and duality

Let V and W be vector spaces.

The space

$$\begin{aligned} \mathrm{Hom}(V,W) &= \mathrm{Hom}_{\mathbb{k}}(V,W) \\ &= \{ T : V \to W \mid T \text{ is linear} \} \end{aligned}$$

is a vector with the operation

$$(a_1 T_1 + a_2 T_2)(\vec{v}) = a_1 T_1(\vec{v}) + a_2 T_2(\vec{v})$$

In particular, one can take $W = \mathbb{k}$.

Def

The dual space of V is the vector space $\mathrm{Hom}(V, \mathbb{k})$, usually denoted V^* or $V^\vee$

Prop

① $\mathrm{Hom}(V, \prod_{\alpha\in I} W_\alpha) \cong \prod_{\alpha\in I} \mathrm{Hom}(V, W_\alpha)$.

② If $\dim V < \infty$, then

$$\mathrm{Hom}(V, \bigoplus_{\alpha\in I} W_\alpha) \cong \bigoplus_{\alpha\in I} \mathrm{Hom}(V, W_\alpha)$$

③ $\mathrm{Hom}(\bigoplus_{\alpha\in I} V_\alpha, W) \cong \prod_{\alpha\in I} \mathrm{Hom}(V_\alpha, W)$

In particular,

$$\left(\bigoplus_{\alpha\in I} V_\alpha\right)^\vee \cong \prod_{\alpha\in I} V_\alpha^\vee$$

Remark

$$\mathrm{Hom}(\bigoplus_{n\in\mathbb{N}} \mathbb{R}, \mathbb{R}) \overset{\text{Prop③}}{\cong} \prod_{n\in\mathbb{N}} \underbrace{\mathrm{Hom}(\mathbb{R}, \mathbb{R})}_{\cong \mathbb{R}} \cong \prod_{n\in\mathbb{N}} \mathbb{R}$$

$$\bigoplus_{n\in\mathbb{N}} \mathrm{Hom}(\mathbb{R}, \mathbb{R}) \cong \bigoplus_{n\in\mathbb{N}} \mathbb{R}$$

Pf. Prop

pf of Prop

$$V \xrightarrow{T} \prod_{\alpha\in I} W_\alpha \xrightarrow{pr_\alpha} W_\alpha$$

①

$$\Phi : \mathrm{Hom}\Big(V, \overset{\bigoplus_{\alpha\in I} W_\alpha}{\prod_{\alpha\in I}} W_\alpha\Big) \to \overset{\bigoplus_{\alpha\in I}}{\prod_{\alpha\in I}} \underbrace{\mathrm{Hom}(V, W_\alpha)}$$

$$\Phi(T) := \sum_{\alpha\in I} pr_\alpha \circ T$$

$pr_\alpha : \prod_{\alpha\in I} W_\alpha \to W_\alpha$

$\sum_{\lambda\in I} \vec{w}_\lambda \mapsto \vec{w}_\alpha$

e.g. $pr_1 : V\oplus W \to V$

$(\vec{v}, \vec{w}) \mapsto \vec{v}$

exer: Φ is linear

$$\Psi : \overset{\bigoplus_{\alpha\in I}}{\prod_{\alpha\in I}} \underbrace{\mathrm{Hom}(V, W_\alpha)}_{T_\alpha} \to \mathrm{Hom}\Big(V, \overset{\bigoplus_{\alpha\in I}}{\prod_{\alpha\in I}} W_\alpha\Big)$$

$$\Psi\Big(\sum_{\alpha\in I} T_\alpha\Big)(\vec{v}) = \sum_{\alpha\in I} \underbrace{T_\alpha(\vec{v})}_{\in W_\alpha} \quad \forall\, \vec{v}\in V$$

exer: $\Phi\circ\Psi = id$, $\Psi\circ\Phi = id$.

② "Use the same formulas"

Note: if $\dim V = \infty$, then Φ is well-defined.

i.e. $\mathrm{im}\,\Phi \not\subseteq \bigoplus_{\alpha\in I} \mathrm{Hom}(V, W_\alpha)$

e.g.

$$id \in \mathrm{Hom}\Big(V, \bigoplus_{n\in\mathbb{N}} \mathbb{R}\Big) \qquad id(e_n) = e_n$$

$$V = \bigoplus_{n\in\mathbb{N}} \mathbb{R}_n \qquad \mathbb{R}_n = \mathbb{R}$$

$$e_n = (0\ldots, \overset{n\text{-th}}{1}, 0\ldots)$$

$\Rightarrow pr_n \circ id : V \to \mathbb{R} : e_n \mapsto 1$

$e_i \mapsto 0$ if $i \neq 0$

$id\circ\iota_n \neq 0 \ \forall n$

$\mathbb{R} \to \bigoplus_{n\in\mathbb{N}} \mathbb{R} : 1 \mapsto e_n$

$$\Rightarrow \sum pr_n \circ id \notin \bigoplus \mathrm{Hom}(V, \mathbb{R})$$

$\Lambda \in \mathbb{N}'$ $\quad n \in \mathbb{N}$

infinite sum

③

$$\tilde{\Phi} : \mathrm{Hom}\left(\bigoplus_{\alpha\in I} V_\alpha, W\right) \longrightarrow \prod_{\alpha\in I} \mathrm{Hom}(V_\alpha, W)$$

$$\tilde{\Phi}(T) := \sum_{\alpha\in I} T \circ \iota_\alpha$$

$\iota_\alpha : V_\alpha \to \bigoplus_{\lambda\in I} V_\lambda$

$V_\alpha \longleftrightarrow \vec{V}_\alpha$

is an iso with the inverse

e.g. $\iota_1 : V \to V \oplus W$

$v \mapsto (v, 0)$

$$\tilde{\Psi} : \prod_{\alpha\in I} \mathrm{Hom}(V_\alpha, W) \longrightarrow \mathrm{Hom}\left(\bigoplus_{\alpha\in I} V_\alpha, W\right)$$

$$\tilde{\Psi}\left(\sum_{\alpha\in I} T_\alpha\right)\left(\sum_{\alpha\in I} \vec{V}_\alpha\right) = \sum_{\alpha\in I} T_\alpha(\vec{V}_\alpha) \in W$$

finite sum $\Rightarrow$ finite sum

#

Prop

If $T : V \to W$ is linear, then one has the induced map

$$T^\vee : W^\vee \longrightarrow V^\vee, \quad T^\vee(\xi) := \xi \circ T$$

where $\quad V \xrightarrow{T} \xi : W \longrightarrow \mathbb{k}$ is linear $\quad (V \xrightarrow{\xi\circ T} \mathbb{k})$

① $T^\vee$ is linear $\left(\because \; (a\xi + b\eta)\circ T = a\xi\circ T + b\eta\circ T = aT^\vee(\xi) + bT^\vee(\eta) \right)$

$T^\vee(a\xi+b\eta)$ "=" $(a\xi+b\eta)\circ T$

② $(\mathrm{id}_V)^\vee = \mathrm{id}_{V^\vee}$

$$(T' \circ T)^\vee = T^\vee \circ (T')^\vee$$

$\because \; (T' \circ T)^\vee(\xi) = \xi \circ (T' \circ T) = (\xi \circ T') \circ T$

$= (T')^\vee(\xi) \circ T = (T^\vee \circ (T')^\vee)(\xi)$

Prop

Let β be a basis for V.

For $\vec{v} \in \beta$, we define

$$\underbrace{\xi_{\vec{v}} : V \to \mathbb{k}}_{\text{linear}}, \quad \xi_{\vec{v}}(\vec{w}) := \begin{cases} 1 & \text{if } \vec{v} = \vec{w} \\ 0 & \text{if } \vec{w} \in \beta,\ \vec{w} \neq \vec{v} \end{cases}$$

① $\{\xi_{\vec{v}} \mid \vec{v} \in \beta\}$ in lin. indep. in $V^\vee$

② If $\dim V < \infty$, then $\{\xi_{\vec{v}} \mid \vec{v} \in \beta\}$ is a basis for $V^\vee$, called the dual basis for $V^\vee$ w.r.t. β.

pf

① If $\underbrace{a_1 \xi_{\vec{v}_1} + \cdots + a_n \xi_{\vec{v}_n}}_{} = 0 : V \to \mathbb{k}$ (any finite $\vec{v}_1, \ldots, \vec{v}_n \in \beta$)

then

$(\qquad\qquad\qquad)(\vec{v}_i) = 0$

$$= a_1 \xi_{\vec{v}_1}(\vec{v}_i) + \cdots + a_n \xi_{\vec{v}_n}(\vec{v}_i)$$

$$= \underline{a_i} \qquad \forall\, i = 1, \cdots, n$$

$\Rightarrow$ lin. indep.

② If $\dim V = n < \infty$, then

$$\dim \mathrm{Hom}(V, \mathbb{k}) = \dim V^\vee = n$$

Also note that

$$\{ \xi_{\vec{v}} \mid \vec{v} \in \beta \}$$

is lin. indep. and has n nonzero vectors

$\Rightarrow$ it is a basis #

We introduce a pairing notation

$$\langle \ \mid \ \rangle : V \times V^\vee \to \mathbb{k}$$

defined by

$$\langle \vec{v} \mid \xi \rangle := \xi(\vec{v}) \qquad \forall\, \vec{v} \in V, \ \xi \in V^\vee$$

<u>Prop</u> The map ..

$\Theta : V \longrightarrow (V^\vee)^\vee$, $\qquad \vec{v} \in V,\ \xi \in V^\vee$

$\Theta(\vec{v})(\xi) = \langle \vec{v} \mid \xi \rangle$

is one-to-one and linear.

If $\dim V < \infty$, then Θ is an iso of vector spaces.

sketch of pf

Recall that if β is a basis for V, then $\tilde{\beta} = \{ \xi_{\vec{v}} \mid \vec{v} \in \beta \}$ is lin. indep.

Note

$$\xi_{\xi_{\vec{v}}}(\eta) = \langle \vec{v}, \eta \rangle \quad \forall\ \eta \in \tilde{\beta}$$

(or solve $\Theta(\vec{v}) = 0$ directly)

$$\langle \vec{v}, \xi \rangle = 0 \quad \forall\ \xi \qquad \cdots$$