Linear Algebra 9/14

§1.1 Vector space and basis

Def

set binary operations

field $\Bbbk = (\Bbbk, +, \cdot)$ satisfies

(1) $a+b = b+a$, $a \cdot b = b \cdot a$

(2) $(a+b)+c = a+(b+c)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(3) $\exists\ 0 \neq 1 \in \Bbbk$ s.t.

$0 + a = a$ $1 \cdot a = a$

(4) $\forall a \in \Bbbk$, $\forall b \neq 0$ in $\Bbbk$, $\exists\ c, d \in \Bbbk$ s.t.

$a + c = 0$ $b \cdot d = 1$

(5) $a \cdot (b+c) = a \cdot b + a \cdot c$

Example

① $\mathbb{R}$ = the field of real numbers

② $\mathbb{C}$ = the field of complex numbers

Def

A vector space (or linear space, $\Bbbk$-vector space) V over a field $\Bbbk$

is $\begin{cases} V & \text{— a set} \\ + & \overset{V\times V\to V}{—} \text{ addition} \\ \cdot & \underset{\mathbb{k}\times V\to V}{—} \text{ scalar product} \end{cases}$] operation

s.t.

(1) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

(2) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

(3) $\exists\ \vec{0} \in \vec{V}$ s.t. $\vec{v}+0 = \vec{v}$

(4) $\forall\ \vec{v} \in \vec{V}\ \exists\ \vec{w} \in V$ s.t. $\vec{v}+\vec{w} = 0$

(5) $\underset{\in \mathbb{k}}{1} \cdot \vec{v} = \vec{v}$

(6) $(ab)\cdot\vec{v} = a\cdot(b\vec{v})$

(7) $a(\vec{v}+\vec{w}) = a\vec{v}+a\vec{w}$

(8) $(a+b)\vec{v} = a\vec{v}+b\vec{v}$

Example

① $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| x_1,\dots,x_n \in \mathbb{R} \right\}$ is a vec. sp. over $\mathbb{R}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

$$a\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} ax_1 \\ \vdots \\ ax_n \end{pmatrix}$$

($\mathbb{k}^n$ is a vec. sp. over $\mathbb{k}$)

② $\mathbb{R}[x] = \{$ polynomials$^{\text{in } x}$ with coeff in $\mathbb{R}\}$

is a vec. sp. over $\mathbb{R}$

③ $\mathbb{k}[x_1, \cdots, x_n] = \{$ poly with variables $x_1, \cdots, x_n$ with coeff in $\mathbb{k}$ $\}$

is a vec. sp. over $\mathbb{k}$

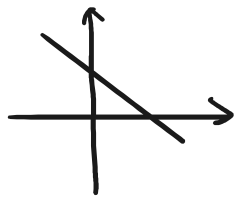
④ $M_{m\times n}(\mathbb{k}) = \{$ $m\times n$ matrices with entries in $\mathbb{k}$ $\}$

is a vec. sp. over $\mathbb{k}$

<u>Remark</u>

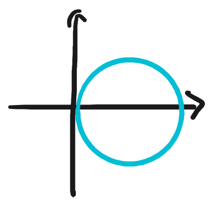
① $\{x+y+1=0\}$ in $\mathbb{R}^2$

is NOT a vector space



② $\{(x-1)^2+y^2=1\}$ in $\mathbb{R}^2$

is NOT a vec. sp.



<u>Def</u> Let V $=(V,+,\cdot)$ be a vec. sp.

(1) A subset W $\neq\phi$ of V is called a <u>subspace</u> if (i) W is closed under $\cdot$ and $+$ ← operations of V

(ii) $\vec{0} \in W$

(2) Let $S \subseteq V$. The space

$$\operatorname{span}(S) := \{\vec{v} \in V \mid \vec{v} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k \text{ for some } a_1, \ldots, a_k \in \mathbb{k},\ \vec{v}_1, \ldots, \vec{v}_k \in V\}$$

is called the subspace spanned (or generated) by S.

smallest subspace of V which contains S

(3) A vector of the form

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$$

is called a linear combination of $\vec{v}_1, \ldots, \vec{v}_k$.

(4) We say $S \subseteq V$ is linearly dependant if $\exists$ distinct $\vec{v}_1, \ldots, \vec{v}_k \in S$ and scalars $a_1, \ldots, a_k \in \mathbb{k}$ NOT all zeros s.t.

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = 0$$

Otherwise, S is linearly independant.

(5) A subset $\beta \subseteq V$ is a basis if

- $\operatorname{span}(\beta) = V$

- span(β) = V
- β is linearly independant

$|\beta|$ = dimension of V = $\dim V$

Thm Let V be a vec. space over $\mathbb{k}$.

(1) Every vector sp. has a basis

(2) $\dim V$ is well-defined. That is, if β_1 and β_2 are two bases for V, then

$$|\beta_1| = |\beta_2|$$

(3) Assume $\dim V < \infty$. A subset $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$ of V is a basis

$\iff \forall \vec{v} \in V, \exists! \, a_1, \cdots, a_n \in \mathbb{k}$ s.t.

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

The unique n-tuple

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{k}^n$$

is called the coordinate of $\vec{v}$ with respect to β, denoted $[\vec{v}]_\beta$

—

Example

① $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th} \quad \in \mathbb{R}^n$

$\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$

called the standard basis for $\mathbb{R}^n$

② $\{1, x, x^2, x^3, \cdots\}$ is a basis for $\mathbb{R}[x]$.

($\Rightarrow$ $\dim \mathbb{R}[x] = \infty$)

③ $\{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid i_1, \dots, i_n$ are nonnegative integers$\}$

is a basis for $\mathbb{k}[x_1, \dots, x_n]$

④ Let $E_{ij} \in M_{m\times n}(\mathbb{k})$ be

$$E_{ij} = \begin{array}{c c} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & 1 & \\ 0 & & & 0 \end{pmatrix} \end{array}$$

$\{E_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $M_{m\times n}(\mathbb{k})$ ($\Rightarrow$ $\dim M_{m\times n}(\mathbb{k}) = m \cdot n$)

Remark Let V be a finite-dim'l vec. sp.

① If $\beta \subseteq V$ is linearly indep and $|\beta| = \dim V$

then β is a basis for V.

② If $\beta^{\subseteq V}$ spans the whole space V and $|\beta| = \dim V$, then β is a basis for V.

Linear map and matrix

Let V, W be vec. sp over $\mathbb{k}$

<u>Def</u>

A map $T : V \to W$ is linear if

$$T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$$

$$\mathrm{Hom}(V, W) = \mathrm{Hom}_{\mathbb{k}}(V, W) = \left\{ \begin{matrix} \mathbb{k}\text{-linear maps} \\ V \to W \end{matrix} \right\}$$

$$\mathrm{End}(V) = \mathrm{Hom}_{\mathbb{k}}(V, V)$$

denoted "$\cong$"

A linear map $T : V \to W$ is called an isomorphism if $\exists$ linear map $T^{-1} : W \to V$ s.t.

$$T^{-1} \circ T = \mathrm{id}_V \ , \quad T \circ T^{-1} = \mathrm{id}_W$$

Two vec. sp. V and W are isomorphic if $\exists$ isomorphism $V \to W$

Note: many "=" mean "is isomorphic to"

Remark

① a linear map $T: V \to W$ is an iso.

$\iff$ T is 1-1 and onto

② the relation "isomorphic" is an equivalence relation, i.e.

- $V \cong V$
- $V \cong W \Rightarrow W \cong V$
- $V \cong W,\ W \cong W' \Rightarrow V \cong W'$

Prop

Let $T, T': V \to W$ be linear, and $\beta = \{\vec{v}_\lambda\}_{\lambda\in\Lambda}$ be a basis for V.

(1) $T = T' \iff T(\vec{v}_\lambda) = T'(\vec{v}_\lambda)\quad \forall\ \vec{v}_\lambda \in \beta$

(2) Given any map $f: \Lambda \to W$, $\exists!$ linear map $T_f: V \to W$ s.t.

$$T_f(\vec{v}_\lambda) = f(\lambda) \in W. \quad \forall\ \lambda \in \Lambda.$$

Thm

$V \cong W \iff \dim V = \dim W$

Def

Let $T : V \to W$ be linear.

(or null space)
kernel of $T = \ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = 0\}$

image of $T = \mathrm{im}(T) = \{\vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V\}$
(or range)

rank of $T = \mathrm{rank}(T) := \dim(\mathrm{im}(T))$

Prop

(1) $\ker(T)$ is a subspace of V

(2) $\mathrm{im}(T)$ " " W

(3) if $\dim V = \dim W$, then

(a) T is an iso

$\Leftrightarrow$ (b) $\ker(T) = 0$

$\Leftrightarrow$ (c) $\mathrm{im}(T) = W$.

Suppose V and W are finite-dim'l.

Let

$\beta = \{\vec{v}_1, \cdots \vec{v}_n\}$: basis for V

$\gamma = \{\vec{w}_1, \cdots \vec{w}_m\}$ basis for W

__Def__

Let $T : V \to W$ be linear.

$\exists!\ a_{ij} \in \mathbb{k}$ s.t.

$$T(\vec{v}_j) = a_{1j}\vec{w}_1 + \cdots + a_{mj}\vec{w}_m$$

$\forall\, j = 1, \cdots, n$. The $m \times n$ matrix

$$[T]_\gamma^\beta := \begin{array}{c} \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & & & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & \cdots & a_{mn} \end{pmatrix} \\ \qquad\quad j \end{array}$$

is the matrix representation of T with respect to β and γ

In the case, $V = W$, $\beta = \gamma$, we also write

$$[T]_\beta = [T]_\beta^\beta$$

__Remark__

Here, β, γ are "ordered bases"

(i.e. the order of vectors in the basis matters when computing matrix repn)

__Thm__

$$\{ T : V \to W \mid T \text{ is linear} \} \longrightarrow M_{m\times n}(\mathbb{k})$$
$$T \longmapsto [T]_\gamma^\beta$$

is an isomorphism of vector spaces
where the operation of linear maps is
$$(aT + bT')(\vec{v}) = aT(\vec{v}) + bT'(\vec{v})$$

<u>Thm</u>

Let V, W, W' be vec. sp. and
$$T : V \to W, \qquad T' : W \to W'$$
be linear.

Let β: basis for V
γ: " W
γ': " W'

Then for $\vec{v} \in V$

matrix multiplication

$$[T(\vec{v})]_\gamma = [T]_\gamma^\beta \cdot [\vec{v}]_\beta$$
$$[T' \circ T]_{\gamma'}^\beta = [T']_{\gamma'}^\gamma \cdot [T]_\gamma^\beta$$

<u>Cor</u>

$T : V \to W$ is an iso $\iff$ $[T]_\gamma^\beta$ is invertible

In this case

In this case,

$$[T^{-1}]_{\beta}^{\gamma} = \left([T]_{\gamma}^{\beta}\right)^{-1}$$

choice of bases give the following corr.

vector space	$\mathbb{k}^n = \{n\times 1 \text{ matrices}\}$
vector	$n\times 1$ matrix (coordinate)
linear map $V\to W$, $\dim W = m$, $\dim V = n$	$m\times n$ matrix
addition	matrix addition
scalar product	scalar product
plugin vector	matrix multiplication
composition	matrix multiplication
iso	invertible matrix
inverse	inverse of matrix

<u>Example</u>

The linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \\ x-y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

has the matrix repn.

w.r.t. the

$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

standard bases

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

because $T(e_1) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} = 1\cdot e_1 + 2\cdot e_2 + 1\cdot e_3$

$T(e_2) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} = 1\cdot e_1 + 0\cdot e_2 + (-1)\, e_3$

Let $\beta' = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ – another basis for $\mathbb{R}^2$

$\Rightarrow T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}$. $\Rightarrow [T]_\gamma^{\beta'} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$.

$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}$

Introduction to Linear Algebra

Hsuan-Yi Liao

國立清華大學
NATIONAL TSING HUA UNIVERSITY

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Advanced Linear Algebra

The theory of Linear Algebra originated from the study of (systems of) linear equations such as

$$\begin{cases} x_1 + 2x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 &= 1 \end{cases} \tag{1}$$

which is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 2 & -5 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

Fundamental questions about an equation

- Does a solution exist?
- If a solution exists, how many solutions does the equation have?
- What are the precise solutions of the equation?

Existence problem for a linear system

Let $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ be the map which takes

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto A\vec{x},$$

$A\vec{x} = \vec{b}$

where $A = \begin{pmatrix} 1 & 2 & -5 \\ 2 & -1 & 0 \end{pmatrix}$ is the coefficient matrix of $\underline{\underline{(2)}}$.

Note that

$$\text{solution of (2) exists} \quad \Leftrightarrow \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathrm{im}(T_A).$$

How many solutions

if (2) has a sol, then $A\vec{x}=\vec{b}$

The solution set of (2) is a parallel transport of the solution set of the corresponding homogeneous equation

$A\vec{x}=\vec{0}$

$$\begin{pmatrix} 1 & 2 & -5 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

whose solution always exists.

$T_A(\vec{x}) = A\vec{x}$

Note that

$$\text{solution set of (3)} = \ker(T_A)$$

which is closed under *scalar product* and *addition*. From here, one can see the importance of vector spaces and linear maps in the study of linear systems.

Linear maps, vectors and linear systems		
linear maps	vectors	linear system
$T_A : \mathbb{R}^n \to \mathbb{R}^m$	column vectors of A	A is an $m \times n$ matrix
$\vec{b} \in \text{im}(T_A)$	$\vec{b} \in \mathrm{span}(\text{columns})$	$A\vec{x} = \vec{b}$ has a sol
T_A is onto	$\mathrm{span}(\text{columns}) = \mathbb{R}^m$	$A\vec{x} = \vec{b}$ has a sol $\forall \vec{b} \in \mathbb{R}^m$
T_A is one-to-one	linear indep columns	$A\vec{x} = \vec{0}$ has a unique sol
$\ker T_A$		sol set of $A\vec{x} = \vec{0}$
$\dim \text{im}\, T_A$	$\dim \mathrm{span}(\text{columns})$	rank A (computable)
$\dim \ker T_A = \cancel{m} - \text{rank}\, A$ n		size of sol set of $A\vec{x} = \vec{0}$

Success of Linear Algebra

Satisfactory answers to the 3 fundamental questions:

- theoretical aspect: the table
- computational aspect: row reduction algorithm

Relation between linear maps and matrices:

- By a basis.
- Matrix techniques apply to linear maps.

Linear Algebra is the foundation of other math theories:

- Advanced Calculus: total differential = linear-map approximation of the map around a point.
  Differentiation: apply Linear Algebra to study local behaviors.
- Differential Geometry: study complicated spaces by vector spaces + gluing techniques
- Abstract Algebra: modules, algebras, etc.
- Applied Math: apply matrices to many problems (e.g. algorithms, probabilities, graphs, etc.)
- and more......

Main topics of the course

Main topic: tensor products and the relevant topics.

Other topics: several important constructions of vector spaces, modules, algebras and matrix-valued functions.

Tensor product:

- important in Algebra, Geometry, Physics, etc.
- universal property = a kind of generalization of basis