Advanced Linear Algebra — Homework 9 (Fall 2022)

Let V and W be vector spaces over a field $\Bbbk$, unless otherwise stated.

1. Show that there exists an inner product on any finite-dimensional vector space over $\mathbb{R}$.
2. Show that there is NO symplectic bilinear form on the one-dimensional vector space $\mathbb{R}$.
3. Show that there is NO symplectic bilinear form on $\mathbb{R}^3$.
4. Let V be finite-dimensional a real vector space. Suppose we are given a **complex structure** J on V, i.e., $J : V \to V$ is a linear map such that
$$J^2 = J \circ J = -\operatorname{id}.$$
   (a) With the complex scalar multiplication
$$(a + bi) \cdot v := a \cdot v + b \cdot J(v), \qquad \forall\, v \in V,$$
   show that V is a complex vector space.
   (b) Show that the dimension of V as a real vector space is even.
   (c) Let ω be a symplectic form on V such that $\omega(Jv, Jw) = \omega(v, w)$ for any $v, w \in V$. Show that the map $g : V \times V \to \mathbb{R}$, defined by $g(v, w) = \omega(v, Jw)$, is a symmetric bilinear form.
   (d) Assume $g(-,-) = \omega(-, J-)$ is positive-definite. (Some people say V is a **Kähler vector space** in this case.) Let $h : V \times V \to \mathbb{C}$ be the map
$$h(v, w) := g(v, w) - i\omega(v, w), \qquad \forall\, v, w \in V.$$
   Regarding V as a complex vector space, show that h is a **Hermitian inner product** on V with the property $h(Jv, Jw) = h(v, w)$, $\forall\, v, w \in V$.
   Recall that $h : V \times V \to C$ is called a **Hermitian inner product** on a complex vector space V if, for any $u, v, w \in V$, any $c \in \mathbb{C}$, one has
      (i) $h(v, w) = \overline{h(w, v)}$,
      (ii) $h(u + v, w) = h(u, w) + h(v, w)$ and $h(u, v + w) = h(u, v) + h(u, w)$,
      (iii) $h(c \cdot v, w) = c \cdot h(v, w)$ and $h(v, c \cdot w) = \bar{c} \cdot h(v, w)$,
      (iv) $h(v, v) \geq 0$, and $h(v, v) = 0$ if and only if $v = 0$.
5. Let A be an abelian group, and n be a positive integer. Show that if $na = 0$ for all $a \in A$, then A is a $\mathbb{Z}_n$-module with the $\mathbb{Z}_n$-action $[k] \cdot a = k \cdot a$. Here $[k] \in \mathbb{Z}_n$ is the image of $k \in \mathbb{Z}$ under the canonical projection $\mathbb{Z} \to \mathbb{Z}_n$.
6. Let M be an R-module over a ring R. If $f : M \to M$ is an R-linear map such that $f \circ f = f$, then
$$M = \ker(f) \oplus \operatorname{im}(f).$$
7. Let M be an R-module, and N be a submodule of M. Show that the equivalence classes $M/N = M/\sim$ of the relation $x \sim y \Leftrightarrow x - y \in N$ is an R-module with the operations
$$a \cdot [x] + b \cdot [y] = [a \cdot x + b \cdot y]$$
   for $a, b \in R$, $x, y \in M$. Here $[x] \in M/N$ is the equivalence class which contains x.
8. Let R be a ring, and N be a submodule of an R-module M.
   (a) Show that the inclusion map $\iota : N \hookrightarrow M$ and the projection map $\pi : M \twoheadrightarrow M/N$ are R-linear maps.
   (b) Show that $\ker \pi = \operatorname{im} \iota$, $\ker \iota = 0$ and $\operatorname{im} \pi = M/N$. That is, the sequence
$$0 \longrightarrow N \xhookrightarrow{\ \iota\ } M \xrightarrow{\ \pi\ } M/N \longrightarrow 0$$
   is a **short exact sequence**.
   (c) Show that if there exists an R-linear map $j : M/N \to M$ such that $\pi \circ j = \operatorname{id}_{M/N}$, then there exists an R-linear map $p : M \to N$ such that $p \circ \iota = \operatorname{id}_N$, and vice versa. (Such an R-linear map is called a **splitting** of the short exact sequence.)
   (d) Show that a splitting described in (c) determines an isomorphism of R-modules
$$M \cong N \oplus M/N.$$
   (e) Find a pair of M and N so that a splitting described in (c) does NOT exist. Justify your answer.