Advanced Linear Algebra — Homework 8 (Fall 2022)

Let V and W be vector spaces over a field $\Bbbk$, unless otherwise stated.

1. Give a detailed proof of the following formula

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

2. Let $T : V \to W$ be a linear map, and k be a positive integer.
   (a) Show that there exists a unique linear map
$$T_* : \Lambda^k V \to \Lambda^k W$$
   such that
$$T_*(v_1 \wedge \cdots \wedge v_k) = T(v_1) \wedge \cdots \wedge T(v_k), \qquad \forall\, v_1, \cdots, v_k \in V.$$
   (b) Assume $T' : W \to W'$ is another linear map. Show that
$$(T' \circ T)_* = T'_* \circ T_*.$$

3. Let $\dim V = n$, and $T : V \to V$ be a linear map. Since $\dim \Lambda^n V = 1$, for any nonzero element $\omega \in \Lambda^n V$, one has
$$T_* \omega = d \cdot \omega$$
for some $d \in \Bbbk$. In this exercise, we would like to show that one can define $\det(T)$ to be d. (An alternative definition of determinant.)
   (a) Show that
$$T_* \omega = \det(T) \cdot \omega, \qquad \forall\, \omega \in \Lambda^n V.$$
   Note that the scalar multiple $\det(T)$ is independent of the choice of ω.
   (b) Let $T' : V \to V$ be another linear map. Show that $\det(T' \circ T) = \det(T') \cdot \det(T)$.
   (c) Show that T is invertible if and only if $\det(T) \neq 0$.

4. Let $\omega : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the map
$$\omega\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) := \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$
   (a) Show that ω is a skew-symmetric bilinear map.
   (b) Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$, and let $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ be its dual basis. Find $a, b, c \in \mathbb{R}$ such that
$$\omega(v, w) = \langle a\, \hat{e}_1 \wedge \hat{e}_2 + b\, \hat{e}_1 \wedge \hat{e}_3 + c\, \hat{e}_2 \wedge \hat{e}_3 \,|\, v \wedge w \rangle, \qquad \forall\, v, w \in \mathbb{R}^3.$$

5. Let V be a 3-dimensional real vector space. Let $\{e_1, e_2, e_3\}$ be a basis for V, and $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ be its dual basis. Let v be the maps
$$\Lambda^1 V^\vee \xrightarrow{\cong} \mathbb{R}^3, \qquad \alpha \mapsto v_\alpha = (a_1, a_2, a_3), \text{ if } \alpha = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3,$$
$$\Lambda^2 V^\vee \xrightarrow{\cong} \mathbb{R}^3, \qquad \omega \mapsto v_\omega = (c_1, c_2, c_3), \text{ if } \omega = c_1\, \hat{e}_2 \wedge \hat{e}_3 + c_2\, \hat{e}_3 \wedge \hat{e}_1 + c_3\, \hat{e}_1 \wedge \hat{e}_2.$$
Show that
$$v_{\alpha \wedge \beta} = v_\alpha \times v_\beta$$
for any $\alpha, \beta \in \Lambda^1 V^\vee$. (The notation $\times$ means the cross product.)

6. Let $\mathcal{F}(V)$ be the set of all the bases for a real n-dimensional vector space V ($n < \infty$), and let $\sim$ be the relation on $\mathcal{F}(V)$ defined by $\{e_i\} \sim \{\epsilon_j\}$ iff $\det\big(a_{ij}\big) > 0$, where a_{ij} are given by $e_i = \sum_{j=1}^n a_{ij} \epsilon_j$. Prove that the relation $\sim$ is an equivalence relation on $\mathcal{F}(V)$ with exactly two equivalence classes.

7. Let V be finite-dimensional, and $T : V \to V$ be an invertible linear map. For a basis $\beta = \{e_1, \cdots, e_n\}$ for V, we denote by $T\beta$ the basis $T\beta = \{e_1, \cdots, e_n\}$.
   (a) Prove the following statements are equivalent.
      (i) There exists a basis β for V such that β and $T\beta$ have the same orientation.
      (ii) For any basis β for V, the bases β and $T\beta$ have the same orientation.
   We say T is **orientation-preserving** if T satisfies one of the above conditions.
   (b) Show that T is orientation-preserving if and only if $\det(T) > 0$.