Advanced Linear Algebra — Homework 7 (Fall 2022)

Let V and W be vector spaces over a field $\Bbbk$.

1. Let $A = \Bbbk[x_1, \cdots, x_k]$ be the space of polynomials in k variables, and let $\mu : A \otimes A \to A$ be the unique linear map such that $\mu(f \otimes g) = fg$ for any $f, g \in A$. Let $X, Y : \Bbbk[x_1, \cdots, x_k] \to \Bbbk[x_1, \cdots, x_k]$ be the linear maps

$$X(f) = a_1 \frac{\partial}{\partial x_1}(f) + \cdots + a_k \frac{\partial}{\partial x_k}(f),$$
$$Y(f) = b_1 \frac{\partial}{\partial x_1}(f) + \cdots + b_k \frac{\partial}{\partial x_k}(f),$$

where $f, a_1, \cdots, a_k, b_1, \cdots, b_k \in A$.

   (a) Show that, for $D = X$ or Y,
$$D \circ \mu = \mu \circ (\mathrm{id}_A \otimes D + D \otimes \mathrm{id}_A) \tag{1}$$
   as maps $A \otimes A \to A$. (Such an operator is called a **derivation**.)

   (b) Show that $D = X \circ Y - Y \circ X$ also satisfies the equation (1).

   (c) Show that there exist derivations X and Y so that $X \circ Y$ does NOT satisfy the equation in (1).

2. Prove or disprove. ($S^0V = \Lambda^0 V = \Bbbk$.)

   (a) $S^2(V \oplus W) \cong S^2V \oplus (V \otimes W) \oplus (V \otimes W) \oplus S^2W$.

   (b) $S^k(V \oplus W) \cong \bigoplus_{i=0}^k (S^iV \otimes S^{k-i}W)$.

   (c) $\Lambda^k(V \oplus W) \cong \bigoplus_{i=0}^k (\Lambda^iV \otimes \Lambda^{k-i}W)$.

3. Let $v, w \in V$. Suppose the base field $\Bbbk = \mathbb{R}$. True or false. Explain your answers.

   (a) $(v + w) \odot (v - w) = v \odot v - w \odot w$.

   (b) $(v + w) \odot (v + w) = v \odot v + 2v \odot w + w \odot w$.

   (c) $(v + 2w) \odot (v + w) = v \odot v + w \odot (2w)$.

   (d) $(2v + w) \odot (v + 3w) = 5 \cdot v \odot w$.

   (e) $(v + w) \wedge (v - w) = v \wedge v - w \wedge w$.

   (f) $(v + w) \wedge (v + w) = 0$.

   (g) $(v + 2w) \wedge (v + w) = v \wedge w$.

   (h) $(2v + w) \wedge (v + 3w) = 5 \cdot v \wedge w$.

4. Recall that $A \in V^{\otimes k}$ is called a symmetric k-tensor if $\tau_\sigma(A) = A$ for all $\sigma \in S_k$. We denote the space of symmetric k-tensors by $\mathrm{Sym}^k V$. Let $\pi : \mathrm{Sym}^k V \to S^kV$ be the restriction of the quotient map.

   (a) Show that there exists a bilinear map $S^kV \times S^lV \to S^{k+l}V$ with the property
$$(v_1 \odot \cdots \odot v_k, v_{k+1} \odot \cdots \odot v_{k+l}) \mapsto v_1 \odot \cdots \odot v_k \odot v_{k+1} \odot \cdots \odot v_{k+l}$$
   for any $v_1, \cdots, v_{k+l} \in V$. This bilinear map is denoted by $\odot$.

   (b) Let $A = v_1 \otimes v_2 + v_2 \otimes v_1$ and $B = v_3$, where $v_1, v_2, v_3 \in V$. Show that $A \in \mathrm{Sym}^2 V$, $B \in \mathrm{Sym}^1 V$, but $A \otimes B \notin \mathrm{Sym}^3 V$.

   (c) Find $C \in \mathrm{Sym}^3 V$ such that
$$\pi(C) = \pi(A) \odot \pi(B).$$

   (d) Show that there exists a bilinear map $\mathrm{Sym}^k V \times \mathrm{Sym}^l V \to \mathrm{Sym}^{k+l} V$, $(A, B) \mapsto A \tilde{\odot} B$ such that
$$\pi(A \tilde{\odot} B) = \pi(A) \odot \pi(B).$$
   Usually, this operation $\tilde{\odot}$ is also denoted by $\odot$.

   (e) Are there analogous operations for Λ^kV?

5. Let $\Bbbk[x_1, \cdots, x_n]^2$ be the space of homogeneous polynomials of degree 2. Suppose $\dim V = n$. Recall that one has isomorphisms

$$\begin{aligned}\phi_1 &: S^2V^\vee \xrightarrow{\cong} \Bbbk[x_1, \cdots, x_n]^2,\\ \phi_2 &: S^2V^\vee \xrightarrow{\cong} \{\text{symmetric bilinear maps } V \times V \to \Bbbk\},\\ \phi_3 &: S^2V^\vee \xrightarrow{\cong} \mathrm{Hom}(S^2V, \Bbbk).\end{aligned}$$

Assume $\xi_1, \xi_2 \in V^\vee$ and $v_1, v_2 \in V$.

(a) Describe ϕ_1, ϕ_2, ϕ_3.

(b) Find a formula for $\phi_2(\xi_1 \odot \xi_2)(v_1, v_2)$.

(c) Find a formula for $\phi_3(\xi_1 \odot \xi_2)(v_1 \odot v_2)$.

(d) Let $\{e_1, \cdots, e_n\}$ be a basis for V and $v \in V$. The evaluation map $\mathrm{ev}_v : \Bbbk[x_1, \cdots, x_n]^2 \to \Bbbk$ is the map

$$\mathrm{ev}_v(f) = f(a_1, \cdots, a_n),$$

for $f \in \Bbbk[x_1, \cdots, x_n]^2$ and $v = \sum_{i=1}^n a_i e_i$. Find a formula for $\mathrm{ev}_v\big(\phi_1(\xi_1 \odot \xi_2)\big)$.

6. Suppose $\dim V = n$. Recall that one has isomorphisms

$$\begin{aligned}\psi_1 &: \Lambda^2V^\vee \xrightarrow{\cong} \{\text{skew-symmetric bilinear maps } V \times V \to \Bbbk\},\\ \psi_2 &: \Lambda^2V^\vee \xrightarrow{\cong} \mathrm{Hom}(\Lambda^2V, \Bbbk).\end{aligned}$$

Assume $\xi_1, \xi_2 \in V^\vee$ and $v_1, v_2 \in V$.

(a) Describe ψ_1 and ψ_2.

(b) Find a formula for $\psi_1(\xi_1 \wedge \xi_2)(v_1, v_2)$.

(c) Find a formula for $\psi_2(\xi_1 \wedge \xi_2)(v_1 \wedge v_2)$.