Advanced Linear Algebra — Homework 6 (Fall 2022)

1. Let $\Bbbk[x_1, \cdots, x_k]^n$ be the vector space of homogeneous polynomials of degree n in k variables, i.e.,

$$\Bbbk[x_1, \cdots, x_k]^n = \operatorname{span}\{x_1^{i_1} x_2^{i_2} \cdots x_k^{i_i} \in \Bbbk[x_1, \cdots, x_k] \mid i_1 + \cdots + i_k = n\}.$$

   (a) Show that $\bigoplus_{p+q=n} \Bbbk[x]^p \otimes \Bbbk[x]^q \cong \Bbbk[x, y]^n$.

   (b) Show that $\Bbbk[x] \otimes \Bbbk[x] \cong \Bbbk[x, y]$.

   (c) Show that $\Bbbk[x_1, \cdots, x_k] \otimes \Bbbk[x_1, \cdots, x_l] \cong \Bbbk[x_1, \cdots, x_{k+l}]$.

2. Let $A = \Bbbk[x_1, \cdots, x_k]$ be the space of polynomials in k variables.

   (a) Show that there exists a unique linear map $\mu : A \otimes A \to A$ such that $\mu(f \otimes g) = fg$ for any $f, g \in A$.

   (b) Let $\Delta^* : \Bbbk[x_1, \cdots, x_{2k}] \to \Bbbk[x_1, \cdots, x_k]$ be the linear map

$$\Delta^*(f)(x_1, \cdots, x_k) = f(x_1, \cdots, x_k, x_1, \cdots, x_k).$$

   Show that there exists an isomorphism $\phi : A \otimes A \xrightarrow{\cong} \Bbbk[x_1, \cdots, x_{2k}]$ such that $\mu = \Delta^* \circ \phi$.

   (c) Show that $\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu)$ as maps $A \otimes A \otimes A \to A$.

   (d) Let $\mu^i : A^{\otimes i+1} \to A$ be the maps defined inductively by

$$\mu^1 = \mu, \qquad \mu^i = \mu \circ (\mu^{i-1} \otimes \mathrm{id}_A).$$

   Show that $\mu \circ (\mu^i \otimes \mu^j) = \mu^{i+j+1}$.

3. Let $\{\xi_1, \cdots, \xi_n\}$ be a linearly independent set in $V^\vee$. Show that there exist $v_1, \cdots, v_n \in V$ such that

$$\xi_i(v_j) = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

4. Let V and W be vector spaces. Recall that there exists a unique linear map

$$\Phi : V^\vee \otimes W \to \mathrm{Hom}(V, W)$$

   which maps $\xi \otimes w$ to $\langle - \mid \xi \rangle \cdot w$, and the map Φ is one-to-one.

   (a) Suppose $\dim V < \infty$. Show that the linear map Φ is an isomorphism.

   (b) Find vector spaces V and W so that the linear map Φ is NOT onto.

5. Suppose $W = V$ in the previous question.

   (a) Show that there exists a unique linear map $E : V^\vee \otimes V \to \Bbbk$ with the property $E(\xi \otimes v) = \xi(v) = \langle v \mid \xi \rangle$.

   (b) Assume V is finite-dimensional. Show that $\mathrm{tr} \circ \Phi = E$ as maps $V^\vee \otimes V \to \Bbbk$, where $\mathrm{tr} : \mathrm{Hom}(V, V) \to \Bbbk$ maps a linear map to its trace. (So one can say "trace $\cong \langle - \mid - \rangle$.")