Advanced Linear Algebra — Homework 5 (Fall 2022)

Suppose V and W are vector spaces over a field $\Bbbk$, unless otherwise stated.

1. Let $x \in V \otimes W$. Show that there exist linearly independent vectors $v_1, \cdots, v_n \in V$ such that
$$x = \sum_{i=1}^{n} v_i \otimes w_i,$$
for some $w_1, \cdots, w_n \in W$.

2. Let $\{e_1, \cdots, e_n\}$ be a basis for V. Show that for each $x \in V \otimes W$, there exist unique $w_1, \cdots, w_n \in W$ such that
$$x = \sum_{i=1}^{n} e_i \otimes w_i.$$

3. Let e_1, e_2 be the standard basis for $\mathbb{R}^2$.
   (a) Find $v, w \in \mathbb{R}^2$ with the property
   $$v \otimes w = 2e_1 \otimes e_1 + 3e_1 \otimes e_2 + 4e_2 \otimes e_1 + 6e_2 \otimes e_2.$$
   (b) Show that
   $$v \otimes w \neq e_1 \otimes e_1 + e_2 \otimes e_2$$
   for any $v, w \in \mathbb{R}^2$.

4. Let V be a real vector space. By considering $\mathbb{C}$ as a 2-dimensional real vector space, one has the tensor product $\mathbb{C} \otimes_{\mathbb{R}} V$.
   (a) Show that any element in $\mathbb{C} \otimes_{\mathbb{R}} V$ is of the form $1 \otimes v_1 + i \otimes v_2$ (or $v_1 + iv_2$ for simplicity), where $i = \sqrt{-1}$.
   (b) Show that $\mathbb{C} \otimes_{\mathbb{R}} V \cong V \oplus V$.
   (c) Show that $\mathbb{C} \otimes_{\mathbb{R}} V$ is a vector space over $\mathbb{C}$ whose scalar multiplication is given by
   $$(x + iy) \cdot (v_1 + iv_2) = (xv_1 - yv_2) + i(yv_1 + xv_2)$$
   for $x, y \in \mathbb{R}$, $v_1, v_2 \in V$. The complex vector space $\mathbb{C} \otimes_{\mathbb{R}} V$ is called the **complexification** of V.
   (d) Show that $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} V)$.
   (e) Let $f : V \to W$ be a $\mathbb{R}$-linear map. Show that there exists a unique $\mathbb{C}$-linear map $\tilde{f} : \mathbb{C} \otimes_{\mathbb{R}} V \to \mathbb{C} \otimes_{\mathbb{R}} W$ such that $\tilde{f}(1 \otimes v) = 1 \otimes f(v)$ for all $v \in V$. The map $\tilde{f}$ is sometimes referred as the **complexification** of f.

5. Let $\tilde{V}$ be a vector space, and $\tilde{\varphi} : V \times W \to \tilde{V}$ be a bilinear map. Suppose $(\tilde{V}, \tilde{\varphi})$ satisfies the universal property: for any bilinear map $f : V \times W \to W'$, there exists a unique linear map $\tilde{f} : \tilde{V} \to W'$ such that $f = \tilde{f} \circ \tilde{\varphi}$. Show that $\tilde{V} \cong V \otimes W$.