Advanced Linear Algebra — Homework 4 (Fall 2022)

1. Recall that an element x in $V \otimes W$ is called a **simple tensor** if $x = v \otimes w$ for some $v \in V$, $w \in W$. Show that a simple tensor $v \otimes w \in V \otimes W$ vanishes in $V \otimes W$ iff $v = 0$ in V or $w = 0$ in W.

2. Let V be a vector space over $\Bbbk$. Construct isomorphisms between V, $V \otimes_{\Bbbk} \Bbbk$ and $\Bbbk \otimes_{\Bbbk} V$. Justify your answers.

3. Let V, W be vector spaces over $\mathbb{R}$, and let $v \in V$ and $w \in W$. Determine the following equations in $V \otimes W$ are true or false. Explain your answers.
   (a) $(v + w) \otimes (v - w) = v \otimes v - w \otimes w$.
   (b) $(v + w) \otimes (v + w) = v \otimes v + w \otimes v + v \otimes w + w \otimes w$.
   (c) $(v + 2w) \otimes (v + w) = v \otimes v + w \otimes (2w)$.
   (d) $(2v + w) \otimes (v + 3w) = 2 \cdot v \otimes v + 6 \cdot v \otimes w + w \otimes v + 3 \cdot w \otimes w$.
   (e) $(6v) \otimes w = (2v) \otimes (3w) = 2 \cdot \big(v \otimes (3w)\big) = 6 \cdot v \otimes w$.
   (f) $v \otimes w + v \otimes w = (2v) \otimes w = v \otimes (2w)$.
   (g) $2 \cdot (v \otimes w) = (2v) \otimes (2w)$.

4. Let $v_1, \cdots, v_n \in V$ and $w_1, \cdots, w_m \in W$ the vectors with the properties
$$V = \operatorname{span}\{v_1, \cdots, v_n\}, \qquad \text{and} \qquad W = \operatorname{span}\{w_1, \cdots, w_m\}.$$
   Prove that
$$V \otimes W = \operatorname{span}\{v_i \otimes w_j \mid i = 1, \cdots, n,\ j = 1, \cdots, m\}.$$

5. Let $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the multiplication map, i.e.,
$$\mu(x, y) = xy,$$
   for any $x, y \in \mathbb{R}$. Prove that there exists a linear map $\tilde{\mu} : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ such that
$$\tilde{\mu}(x \otimes y) = \mu(x, y),$$
   for any $x, y \in \mathbb{R}$.