# Advanced Linear Algebra — Homework 3 (Fall 2022)

1. What is the shape (or dimension) of the solution set of the following linear system?

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ 4x_1 + 5x_2 + 6x_3 - x_4 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= 0 \end{cases}$$

2. Let $W$ be a subspace of $V$. Then we have the inclusion map $\iota : W \hookrightarrow V$ and the projection map $\pi : V \twoheadrightarrow V/W$.

   (a) Show that $\ker \pi = \operatorname{im} \iota$, $\ker \iota = 0$ and $\operatorname{im} \pi = V/W$. That is, the sequence

$$0 \longrightarrow W \overset{\iota}{\hookrightarrow} V \overset{\pi}{\twoheadrightarrow} V/W \longrightarrow 0$$

   is a **short exact sequence**.

   (b) Show that there exists a linear map $j : V/W \to V$ such that $\pi \circ j = \mathrm{id}_{V/W}$.

   (c) Show that there exists a linear map $p : V \to W$ such that $p \circ \iota = \mathrm{id}_W$.

   (d) Show that a choice of $j : V/W \to V$, $\pi \circ j = \mathrm{id}_{V/W}$, is equivalent to a choice of $p : V \to W$, $p \circ \iota = \mathrm{id}_W$. (Such a choice is called a **splitting** of the short exact sequence.)

   (e) Show that a choice of $j : V/W \to V$, $\pi \circ j = \mathrm{id}_{V/W}$, (or, equivalently, a choice of $p : V \to W$, $p \circ \iota = \mathrm{id}_W$) determines an isomorphism

$$V \overset{\cong}{\longrightarrow} W \oplus V/W, \quad v \mapsto \big(p(v), \pi(v)\big).$$

   In particular, $V \cong W \oplus V/W$, and $\dim(V/W) = \dim V - \dim W$.

3. Let $V_1, \cdots, V_k$ and $W$ be vector spaces. Prove that $k$-linear maps $V_1 \times \cdots \times V_k \to W$ forms a vector space by the operation

$$(af + bg)(v_1, \cdots, v_k) = af(v_1, \cdots, v_k) + bg(v_1, \cdots, v_k).$$

4. Let $V_1, V_2, V_3$ and $W$ be finite-dimensional vector spaces. Let

$$\beta_p = \{v_1^p, \cdots, v_{n_p}^p\} \quad \text{and} \quad \gamma = \{w_1, \cdots, w_m\}$$

   be bases for $V_p$ and $W$, respectively.

   (a) Show that a multilinear map

$$f : V_1 \times V_2 \times V_3 \to W$$

   is uniquely determined by the vectors

$$f(v_{i_1}^1, v_{i_2}^2, v_{i_3}^3), \qquad 1 \le i_p \le n_p,\ p = 1, 2, 3,$$

   in $W$.

   (b) Assume $f_{i_1,i_2,i_3}^j \in \Bbbk$ are the numbers such that

$$f(v_{i_1}^1, v_{i_2}^2, v_{i_3}^3) = \sum_{j=1}^m f_{i_1,i_2,i_3}^j w_j.$$

   Show that the map

$$\begin{aligned} \Big\{\text{multilinear maps } V_1 \times V_2 \times V_3 \to W\Big\} &\longrightarrow \left\{ \left(a_{i_1,i_2,i_3}^j\right)_{\substack{1 \le j \le m \\ 1 \le i_p \le n_p}} \,\middle|\, a_{i_1,i_2,i_3}^j \in \Bbbk \right\}, \\ f &\longmapsto \left(f_{i_1,i_2,i_3}^j\right)_{\substack{1 \le j \le m \\ 1 \le i_p \le n_p}}, \end{aligned}$$

   is an isomorphism of vector spaces.