Advanced Linear Algebra — Homework 2 (Fall 2022)

1. Let β_α be a basis for V_α, $\alpha \in I$. Show that the disjoint union $\beta = \coprod_{\alpha \in I} \beta_\alpha$ is a basis for the direct sum $\bigoplus_{\alpha \in I} V_\alpha$.

2. Let I be an arbitrary set, and $V_\alpha = \Bbbk$ for any $\alpha \in I$. Show that $\bigoplus_{\alpha \in I} V_\alpha = \Bbbk^{(I)}$, where $\Bbbk^{(I)}$ is the $\Bbbk$-vector space freely generated by I.

3. Let $\mathbb{N}$ be the set of positive integers, and $V_n = \mathbb{R}$ for any $n \in \mathbb{N}$. Show that a basis for $\bigoplus_{n \in \mathbb{N}} V_n = \bigoplus_{n \in \mathbb{N}} \mathbb{R}$ is countable, but a basis for $\prod_{n \in \mathbb{N}} V_n = \prod_{n \in \mathbb{N}} \mathbb{R}$ is uncountable.

4. Show that there exist vector spaces W and V_α, $\alpha \in I$, such that the linear map

$$\Psi : \bigoplus_{\alpha \in I} \mathrm{Hom}(W, V_\alpha) \to \mathrm{Hom}\Big(W, \bigoplus_{\alpha \in I} V_\alpha\Big), \quad \Psi\Big(\sum_{\alpha \in I} T_\alpha\Big)(w) = \sum_{\alpha \in I} T_\alpha(w)$$

   is not onto.

5. Find a vector space V and a basis β for V with the property

$$\mathrm{span}\,\{\xi_v \mid v \in \beta\} \neq V^\vee,$$

   where $\xi_v : V \to \Bbbk$ is the linear map characterized by the property

$$\xi_v(w) = \begin{cases} 1, & \text{if } w = v, \\ 0, & \text{if } w \neq v, w \in \beta. \end{cases}$$

   Justify your answer.

6. Let V be a vector space, and

$$\theta : V \to (V^\vee)^\vee, \quad \theta(v)(\xi) := \langle v \,|\, \xi \rangle = \xi(v).$$

   (a) Prove that θ is linear and one-to-one.
   (b) Prove that if $\dim V < \infty$, then θ is an isomorphism.
   (c) Show that θ is not an isomorphism if $V = \bigoplus_{n \in \mathbb{N}} \mathbb{R}$.