Advanced Linear Algebra — Homework 1 (Fall 2022)

1. Let V be a vector space over a field $\Bbbk$. Prove that a subset $\beta = \{\vec{v}_\lambda\}_{\lambda\in\Lambda}$ of V (not necessarily a finite set) is a basis for V if and only if for each $\vec{v} \in V$, there exists a unique function $\Lambda \to \Bbbk : \lambda \mapsto a_\lambda$ such that
   - $a_\lambda \neq 0$ only for a finite number of indexes $\lambda \in \Lambda$, and
   - $\vec{v} = \sum_{\lambda\in\Lambda} a_\lambda \vec{v}_\lambda$.

2. Let $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\gamma = \{\vec{w}_1, \cdots, \vec{w}_n\}$ be two bases for a vector space V. Show that there exists an invertible matrix A such that
$$[\vec{v}]_\gamma = A[\vec{v}]_\beta$$
for all $\vec{v} \in V$.

3. Let $T : V \to W$ be a linear map between vector spaces. Prove that T is an isomorphism if and only if T is one-to-one and onto.

4. Prove that the relation "isomorphic" is an equivalence relation.

5. Let $T, T' : V \to W$ be linear maps between vector spaces, and $\beta = \{\vec{v}_\lambda\}_{\lambda\in\Lambda}$ be a basis for V.
   (a) Show that $T = T'$ if and only if $T(\vec{v}_\lambda) = T'(\vec{v}_\lambda)$ for all $\lambda \in \Lambda$.
   (b) Given an arbitrary map $f : \Lambda \to W : \lambda \mapsto \vec{w}_\lambda$ (not necessarily linear), prove that there exists a unique linear map $T_f : V \to W$ such that $T_f(\vec{v}_\lambda) = \vec{w}_\lambda$ for all $\lambda \in \Lambda$.
   (c) Prove that two vector spaces V and W are isomorphic if and only if $\dim V = \dim W$.

6. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{w}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.
   (a) Show that $\beta = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$.
   (b) Show that $\gamma = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a basis for $\mathbb{R}^3$.
   (c) Find the matrix representation $[T]_\gamma^\beta$ of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$,
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \\ x-y \end{pmatrix}$$
   with respect to β and γ.
   (d) Find invertible matrices $A \in \mathrm{GL}_2(\mathbb{R})$, $B \in \mathrm{GL}_3(\mathbb{R})$ such that
$$[T]_\gamma^\beta = B\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} A.$$

7. Let
$$V_n = \{p(x) \in \mathbb{R}[x] \mid p(x) \text{ is of the form } a_0 + a_1 x + \cdots + a_n x^n\}.$$
Let $a \in \mathbb{R}$ be a fixed number, and let
$$D_n : V_n \to V_{n-1},\ D_n(p) := \frac{dp}{dx},$$
$$I_{a,n} : V_n \to V_{n+1},\ I_{a,n}(p) := \int_a^x p(t)\,dt,$$
$$I_n' : V_n \to V_{n+1},\ I_n'\Big(\sum_{i=0}^n a_i x^i\Big) := 1 + \sum_{i=0}^n \frac{a_i}{i+1}\,x^{i+1}.$$
   (a) Show that, for each nonnegative integer n, V_n is a vector space of dimension $n+1$.
   (b) Show that, for each nonnegative integer n, D_n and $I_{a,n}$ are linear, but I_n' is not.
   (c) Are D_n and $I_{a,n}$ isomorphisms? Are $D_n, I_{a,n}$ and I_n' one-to-one or onto? Explain your answers.
   (d) Let $\beta_n = \{1, x, \cdots, x^n\}$ be the standard basis for V_n. Find the matrices $[D_n]_{\beta_{n-1}}^{\beta_n}$ and $[I_{a,n}]_{\beta_{n+1}}^{\beta_n}$.

(e) Show that

$$\gamma_n = \{1, 1+x, 1+x+x^2, \cdots, 1+x+\cdots+x^n\}$$

is a basis for V_n.

(f) Find the matrix A_n such that

$$[p]_{\gamma_n} = A_n[p]_{\beta_n}$$

for any $p \in V_n$.

(g) Find the matrix representations $[D_n]_{\beta_{n-1}}^{\gamma_n}$, $[D_n]_{\gamma_{n-1}}^{\gamma_n}$, and $[I_{a,n}]_{\gamma_{n+1}}^{\beta_n}$.

(h) Find the matrix representations $[D_{n+1} \circ I_{a,n}]_{\gamma_n}^{\beta_n}$, $[D_{n+1} \circ I_{a,n}]_{\beta_n}^{\gamma_n}$ and $[I_{a,n-1} \circ D_n]_{\gamma_n}^{\gamma_n}$.