Advanced Linear Algebra — Homework 12 (Fall 2022)

1. Let V be a vector space over $\mathbb{R}$. Suppose V is equipped with an inner product $\langle -, - \rangle$. Let $\|v\| = \sqrt{\langle v, v \rangle}$ for $v \in V$. Show that $\| - \|$ is a norm on V with the property

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

   This equality is called the **parallelogram law**.

2. Let V be a normed vector space over $\mathbb{R}$ with the norm $\| - \|$. Assume the parallelogram law is satisfied. Show that there exists an inner product $\langle -, - \rangle$ on V such that $\|v\| = \sqrt{\langle v, v \rangle}$.

3. Find a norm which is not induced by an inner product.

4. Prove that the operator norm

$$\|A\|_{\mathrm{op}} = \sup_{\substack{\|v\|=1, \\ v \in \mathbb{C}^n}} \|Av\|$$

   is a norm on $\mathrm{M}_n(\mathbb{C})$ with the property

$$\|AB\|_{\mathrm{op}} \le \|A\|_{\mathrm{op}} \cdot \|B\|_{\mathrm{op}}, \qquad \forall\, A, B \in \mathrm{M}_n(\mathbb{C}).$$

5. Compute $\exp(tA)$, where
   (a) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.
   (b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
   (c) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

6. Find the general solution of the differential equation

$$y^{(4)} - 10y'' + 25y = 0.$$

7. Let $A \in \mathrm{M}_n(\mathbb{C})$. Show that $\det(e^A) = e^{\mathrm{tr}(A)}$.