Advanced Linear Algebra — Homework 11 (Fall 2022)

bonus = won't be in the exam. And you can write a bonus report about the questions marked with bonus. If there are many people write about the same question, then the points you get will be divided by the number of people writing the same question.

1. An ideal I of an R-algebra A is called a **maximal ideal** if for any ideal J with $I \subset J$, either $J = I$ or $I = A$. An **algebra morphism** f from an R-algebra A to another R-algebra A' is a map $f : A \to A'$ such that $f(r \cdot a) = r \cdot f(a)$, $f(ab) = f(a)f(b)$, $f(a+b) = f(a) + f(b)$, and $f(1_A) = f(1_{A'})$, for any $r \in R$, $a, b \in A$.
   (a) Show that if $f : A \to A'$ is an algebra morphism, then $\ker(f) = f^{-1}(0)$ is an ideal of A.
   (b) Show that for any proper (i.e. $I \neq A$) ideal I of A, there exists an algebra morphism $f : A \to A'$ such that $I = \ker(f)$.
   (c) Suppose I is an ideal of a $\mathbb{k}$-algebra A ($\mathbb{k}$ is a field) such that $A/I \cong \mathbb{k}$. Show that I is a maximal ideal.
   (d) Suppose A is a commutative $\mathbb{k}$-algebra. Prove or disprove that an ideal I of A is a maximal ideal if and only if $R/I \cong \mathbb{k}$.

2. Let $C^\infty(\mathbb{R})$ be the algebra of real-valued infinitely differentiable functions on $\mathbb{R}$.
   (a) Given any $x \in \mathbb{R}$, one has the evaluation map $\mathrm{ev}_x : C^\infty(\mathbb{R}) \to \mathbb{R}$ defined by $\mathrm{ev}_x(f) = f(x)$. Show that ev_x is an algebra morphism.
   (b) Show that for any $x \in \mathbb{R}$, $\ker(\mathrm{ev}_x)$ is a maximal ideal of $C^\infty(\mathbb{R})$.
   (c) (bonus) Show that for every algebra morphism $f : C^\infty(\mathbb{R}) \to \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f = \mathrm{ev}_x$. Therefore, there is a one-to-one correspondence between

$$\mathbb{R} \longleftrightarrow \{\text{algebra morphisms } C^\infty(\mathbb{R}) \to \mathbb{R}\} \longleftrightarrow \{\text{maxinal ideals of } C^\infty(\mathbb{R})\}.$$

   In particular, the space $\mathbb{R}$ is completely determined by the algebra $C^\infty(\mathbb{R})$, and in fact, $\mathbb{R}$ can be replaced by $\mathbb{R}^n$ here.

3. Let $C^\infty(\mathbb{R})$ be the algebra of real-valued infinitely differentiable functions on $\mathbb{R}$.
   (a) For $x \in \mathbb{R}$, we say X is a **derivation of** $C^\infty(\mathbb{R})$ **at** x if X is a linear map $X : C^\infty(\mathbb{R}) \to \mathbb{R}$ such that $X(fg) = X(f)g(x) + f(x)X(g)$ for any $f, g \in C^\infty(\mathbb{R})$. Show that for each $x \in \mathbb{R}$, the derivations of $C^\infty(\mathbb{R})$ at x form a one-dimensional real vector space.
   (b) Show that if $X : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a derivation of $C^\infty(\mathbb{R})$, then the map

$$X|_x : C^\infty(\mathbb{R}) \to \mathbb{R},\ X|_x(f) = X(f)(x)$$

   is a derivation of $C^\infty(\mathbb{R})$ at x.
   (c) (bonus) Show that if $X : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a derivation of $C^\infty(\mathbb{R})$, then there exists $a \in C^\infty(\mathbb{R})$ such that

$$X(f) = a\frac{d}{dx}(f), \qquad \forall\, f \in C^\infty(\mathbb{R}).$$

   (d) Compute the Lie bracket $[a\frac{d}{dx}, b\frac{d}{dx}]$ for $a, b \in C^\infty(\mathbb{R})$.

4. Let $\mathfrak{g}$ be a vector space over $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$. Let $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a bilinear map.
   (a) Show that the condition $[x,x] = 0$ for all $x \in \mathfrak{g}$ is equivalent to the condition $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$.
   (b) Suppose the bilinear map $[-,-]$ satisfies the condition in (a). Show that $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ is equivalent to $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$.

5. Let A be an algebra over a field $\mathbb{k}$. Prove that A endowed with the commutator is a Lie algebra.

6. Let $\mathfrak{g}$ be a Lie algebra whose Lie bracket is the zero bracket $[x,y] = 0$, $\forall\, x, y \in \mathfrak{g}$. What is the universal enveloping algebra $\mathcal{U}\mathfrak{g}$?

7. Let $\mathfrak{g}$ be a Lie algebra. Construct a linear map $f : S\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ such that $f|_{S^k\mathfrak{g}} \neq 0$ for any k.

8. Let A be an algebra over a field $\mathbb{k}$. Show that there is a one-to-one correspondence between A-modules and (algebra) representations of A.

9. Let $\mathfrak{g}$ be a Lie algebra. Show that there is a one-to-one correspondence between (Lie algebra) representations of $\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}$-modules.

10. Suppose $\gamma : \mathbb{R} \to \mathrm{M}_{m\times n}(\mathbb{C})$ is differentiable. Show that the limit $\lim_{h\to 0} \dfrac{\gamma(t+h)-\gamma(t)}{h}$ is same as the derivative $\gamma'(t)$ defined in class.