Advanced Linear Algebra — Homework 10 (Fall 2022)

Let V and W be vector spaces over a field $\Bbbk$.

1. Let M be a module over a commutative ring R.
   (a) For $x \in M$, show that
   $$I_x = \{r \in R \mid r \cdot x = 0\}$$
   is an ideal of R. If $I_x \neq 0$, x is said to be a **torsion element** of M.
   (b) Suppose R is an **integral domain**, i.e., $rs = 0$ in R implies $r = 0$ or $s = 0$. Show that the set T of torsion elements of M is a submodule of M. (T is called the **torsion submodule**.)
   (c) Show that (b) may be false if R is not an integral domain.

2. Let M be a module over a commutative ring R.
   (a) Show that any intersection of submodules of M is still a submodule of M.
   (b) Let S be an arbitrary subset of M. Let N be the intersection of all the submodules of M which contains S. Show that S generates N and that N is the smallest submodule containing S. This submodule N is called the **submodule generated by** S.

3. Let $\mathbb{Z}_n$ be the quotient $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Q}$ be the $\mathbb{Z}$-module of rational numbers.
   (a) Show that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_c$, where c is the greatest common factor of m and n.
   (b) Let M be a $\mathbb{Z}$-module whose torsion submodule is itself M. Show that $M \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.
   (c) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

4. State the definition of quotient algebra. Show that the quotient algebra is an algebra, and the natural map $A \twoheadrightarrow A/I$ is a surjective algebra morphism.

5. Show that any intersection of ideals of an algebra is still an ideal.

6. Recall that the symmetric algebra SV is the quotient algebra
   $$SV = TV \Big/ \langle \{x \otimes y - y \otimes x \mid x, y \in V\} \rangle.$$
   We denote by $S^k V$ the image of $T^k V$ under the quotient map and by $\odot$ the induced product in SV.
   (a) Show that
   $$S^k V = \overbrace{V \otimes \cdots \otimes V}^{k \text{ times}} \Big/ \operatorname{span}\{v_1 \otimes \cdots \otimes (v_i \otimes v_{i+1} - v_{i+1} \otimes v_i) \otimes \cdots v_k\}.$$
   (b) Show that $SV = \bigoplus_{k=0}^{\infty} S^k V$.

7. Construct an isomorphism $\Bbbk[x_1, \cdots, x_n] \cong S(\Bbbk^n)^\vee$ which sends the subspace of polynomials of degree k to $S^k(\Bbbk^n)^\vee$.

8. Recall the exterior algebra ΛV is the quotient algebra
   $$\Lambda V = TV \Big/ \langle \{x \otimes y + y \otimes x \mid x, y \in V\} \rangle.$$
   We denote by $\Lambda^k V$ the image of $T^k V$ under the quotient map and by $\wedge$ the induced product in ΛV.
   (a) Show that
   $$\Lambda^k V = \overbrace{V \otimes \cdots \otimes V}^{k \text{ times}} \Big/ \operatorname{span}\{v_1 \otimes \cdots \otimes (v_i \otimes v_{i+1} + v_{i+1} \otimes v_i) \otimes \cdots v_k\}.$$
   (b) Show that $\Lambda V = \bigoplus_{k=0}^{\infty} \Lambda^k V$.
   (c) Show that if $\dim V = n$, then $\dim \Lambda V = 2^n$.

9. Suppose $\dim V < \infty$. Let α be a nonzero element in $\Lambda^1 V$, and $\gamma \in \Lambda^k V$. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $\beta \in \Lambda^{k-1} V$.