Prop (p.95)

Suppose $|\alpha|<1$ and

$$B_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha} z}.$$

Note: $B_\alpha(\alpha)=0$

Then $B_\alpha : D=D(0;1) \longrightarrow D$ is analytic

pf

Note $1-\bar{\alpha}z=0 \Leftrightarrow z=\frac{1}{\bar{\alpha}}$ and $\left|\frac{1}{\bar{\alpha}}\right| = \frac{1}{|\alpha|} > 1 \Rightarrow B_\alpha$ is analytic in D.

If $|z|=1$, then

$$|B_\alpha(z)|^2 = B_\alpha(z)\overline{B_\alpha(z)} = \frac{z-\alpha}{1-\bar{\alpha}z}\cdot\frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}} = \frac{z\bar{z}-z\bar{\alpha}-\alpha\bar{z}+\alpha\bar{\alpha}}{1-\bar{\alpha}z-\alpha\bar{z}+|\alpha|^2|z|^2}$$

$$= \frac{1+|\alpha|^2-(z\bar{\alpha}+\bar{z}\alpha)}{1+|\alpha|^2-(\bar{\alpha}z+\alpha\bar{z})} = 1$$

By Maximum Modulus Thm, $|B_\alpha(z)|<1 \quad \forall |z|<1$. #

Example (p.95-96)

① Suppose f is analytic and bounded by 1 in the unit disc and $f(\frac{1}{2})=0$.

Show that $|f(\frac{3}{4})| \le \frac{2}{5}$ and "=" is achieved by some analytic function.

pf

Let

$$g(z) := \begin{cases} f(z)/B_{\frac{1}{2}}(z) = \underbrace{f(z)}_{|\cdot|\le 1} \Big/ \underbrace{\left(\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right)}_{|\cdot|=1 \text{ when } |z|=1} & z\neq\frac{1}{2} \\ \frac{3}{4}f'(\frac{1}{2}) = \lim_{z\to\frac{1}{2}} f(z)/B_{\frac{1}{2}}(z) & z=\frac{1}{2} \end{cases}$$

$\Rightarrow g(z)$ is analytic and $\overline{\lim_{|z|\to 1}} |g(z)| \le 1 \overset{\text{Max Mod Thm}}{\Longrightarrow} |g(z)|\le 1 \quad \forall |z|<1$

$\Rightarrow |f(z)| \le |B_{\frac{1}{2}}(z)| = \left|\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right|$

In particular, $|f(\frac{3}{4})| \le |B_{\frac{1}{2}}(\frac{3}{4})| = \frac{2}{5}$

Note that "=" is achieved by $f(z)=B_{\frac{1}{2}}(z)$ #

② Suppose $f: D=D(0;1)\to D$ is analytic s.t.

$$|f'(\tfrac{1}{3})| = \max\{|h'(\tfrac{1}{3})| : h: D\to D \text{ is analytic}\}$$

Show that $f(\frac{1}{3})=0$

pf

Suppose $f(\frac{1}{3})\neq 0$ and consider $g(z) = \dfrac{f(z)-f(\frac{1}{3})}{1-\overline{f(\frac{1}{3})}f(z)} = B_{f(\frac{1}{3})}(f(z))$

We have $g: D=D(0;1) \xrightarrow{f} D \xrightarrow{B_\alpha,\ \alpha=f(\frac{1}{3})} D$

In particular, $|g(z)|<1 \quad \forall |z|<1$.

Note that

$$|g'(\tfrac{1}{3})| = \left|\frac{\cancel{f'(\frac{1}{3})(1-\overline{f(\frac{1}{3})}f(\frac{1}{3}))} - \overbrace{\cancel{(f(\tfrac{1}{3})-f(\tfrac{1}{3}))}}^{=0}(-\overline{f(\frac{1}{3})}f'(\frac{1}{3}))}{(1-\overline{f(\frac{1}{3})}f(\frac{1}{3}))^{\cancel{2}}}\right| = \left|\frac{f'(\frac{1}{3})}{1-\underbrace{|f(\frac{1}{3})|^2}_{\neq 0 \text{ by assumption}}}\right| > \overset{\text{max by assumption}}{|f'(\tfrac{1}{3})|} \ (\rightarrow\leftarrow)$$ #

exer (exer 10, 11 in Ch7)

max $|f'(\frac{1}{3})|$ is assumed by $B_{\frac{1}{3}}(z)$.

Converse of rectangle thm

Recall (Rectangle Thm i.e. Thm 6.1)

Suppose f is analytic in $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $R\subseteq U$ is a rectangle. Then

$$\int_\Gamma f(z)\,dz = 0$$

where $\Gamma=\partial R$.

Morera Thm (Thm 7.4)

Let f be a continuous function on an open set $U\subseteq\mathbb{C}$. If

$$\int_\Gamma f(z)\,dz = 0$$

whenever Γ is the boundary of a closed rectangle in U, then f is analytic in U.

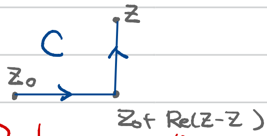
pf

Let $z_0 \in U$, $\varepsilon > 0$ s.t. $D(z_0;\varepsilon) \subseteq U$. Let

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta = \int_C f(\zeta)\,d\zeta$$

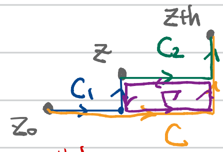
Same as the proof of Integral Thm (Thm 4.15)

where $z \in D(z_0;\varepsilon)$ and C is the curve



By assumption

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\left(\int_z^{z+h} f(\zeta)\,d\zeta + \underbrace{\int_\Gamma f(\zeta)\,d\zeta}_{\substack{\| \\ 0 \text{ by assumption}}}\right)$$



$$\Rightarrow \left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \frac{1}{|h|}\left|\int_z^{z+h} f(\zeta)-f(z)\,dz\right| \le \frac{1}{|h|}\cdot \underbrace{\text{length}(C_2)}_{\le 2|h|}\cdot\left(\sup_{\zeta\in C_2}|f(\zeta)-f(z)|\right)$$

$$\le 2\cdot\left(\sup_{\zeta\in C_2}|f(\zeta)-f(z)|\right) \longrightarrow 0 \quad \text{as } h\to 0 \quad (\text{by continuity of } f)$$

So F is analytic in $D(z_0;\varepsilon)$ and $F'(z) = f(z)$ in $D(z_0;\varepsilon)$

$\Rightarrow f$ is analytic at z_0

Since z_0 is arbitrary in U, we conclude f is analytic in U. #

Def 7.5

Suppose $\{f_n\}$ and f are defined in D. We say f_n converges to f uniformly on compacta if $f_n \to f$ uniformly on every compact subset $K \subseteq D$.

Thm 7.6

Suppose $\{f_n\}$ is a seq of functions, analytic in an open domain D and $f_n \to f$ uniformly on compacta. Then f is analytic in D

pf

Given any $z_0 \in D$, $\exists\, \varepsilon > 0$ s.t. $\overline{D(z_0;\varepsilon)} \subseteq D$.

Since $\overline{D(z_0;\varepsilon)}$ is compact, $f_n \to f$ uniformly in $D(z_0;\varepsilon) =: U$

Since f_n are continuous, f is also continuous in U.

Let Γ be the boundary of any rectangle in U.

$$\int_\Gamma f(z)\,dz = \int_\Gamma \lim_{n\to\infty} f_n(z)\,dz \overset{\text{conv unif in } U}{=} \lim_{n\to\infty}\int_\Gamma f_n(z)\,dz \overset{\text{rectangle thm}}{=} 0$$

By Morera's thm, f is analytic in $U \Rightarrow f$ is analytic at z_0 $\forall z_0 \in D$. #

Example (p.98-99)

The function

$$f(z) := \int_0^\infty \frac{e^{zt}}{t+1}\,dt$$

is analytic in the left half plane $U = \{z\in\mathbb{C} : \operatorname{Re} z < 0\}$

pf

Let $f_n(z) = \int_0^n \frac{e^{zt}}{t+1}\,dt$. Let Γ be the boundary of an arbitrary rectangle in U.

By Fubini's thm,

$$\int_\Gamma\int_0^n \frac{e^{zt}}{t+1}\,dt\,dz \overset{\text{expand by def. apply Fubini to Re and Im respectively}}{=} \int_0^n\int_\Gamma \frac{e^{zt}}{t+1}\,dz\,dt \overset{\text{Rectangle Thm}}{=} \int_0^n 0\,dt = 0$$

$\leftarrow$ entire for each $t \ge 0$

By Morera Thm, f_n is analytic in U $\forall n$. Furthermore

$$|f_n(z)-f(z)| = \left|\int_n^\infty \frac{e^{zt}}{t+1}\,dt\right| \le \int_n^\infty |e^{zt}|\,dt = \int_n^\infty e^{t\cdot Re(z)}\,dt = \frac{1}{Re(z)}e^{t\cdot Re(z)}\Big|_{t=n}^{\infty} = \frac{-1}{Re(z)}e^{\underbrace{n\cdot Re(z)}_{<0}}$$

For any compact $K \subseteq U$, $\exists\, M > 0$ s.t. $Re(z) \le -M$ $\forall z \in K$

$$\Rightarrow |f_n(z)-f(z)| \le \frac{-1}{Re(z)}e^{n\cdot Re(z)} \le \frac{1}{M}\cdot(e^M)^{-n} \longrightarrow 0 \quad \text{as } n\to\infty$$

$\leftarrow$ independent of z

So $f_n \to f$ uniformly in K.

By Thm 7.6, f is analytic in U. #

Thm 7.7

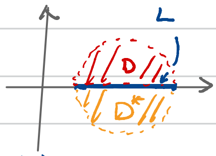
Suppose f is continuous in an open set D and analytic in D except possibly at the points of a line segment. Then f is analytic throughout D.

pf: Apply Morera Thm. Skip.

Reflection principle

Let D be a region which is contained in the upper or lower half plane.

Suppose ∂D contains a segment L on the real axis



Thm 7.8 (Schwarz Reflection Principle)

Suppose f is continuous in $\bar{D}$ and analytic in D (called "C-analytic" in book)

and $f(L) \subseteq \mathbb{R}$.

Then the function

$$g(z) := \begin{cases} f(z), & z \in D \cup L \\ \overline{f(\bar{z})}, & z \in D^* \end{cases}$$

is analytic in $D \cup L \cup D^*$, where $D^* = \{ z \in \mathbb{C} : \bar{z} \in D \}$

pf

① Since $g = f$ in D, g is analytic in D

② If $z \in D^*$, $\forall h$ s.t. $z+h \in D^*$, we have

$$\frac{g(z+h) - g(z)}{h} = \overline{\left(\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}} \right)} \longrightarrow \overline{f'(\bar{z})} \qquad \text{as } h \to 0$$

$\Rightarrow$ g is analytic at z, $\forall z \in D^*$

③ Since f is continuous and real on L, g is also continuous on L

So, by Thm 7.7, g is analytic throughout $D \cup L \cup D^*$ #

Cor 7.9

If f is analytic in a region symmetric with respect to the real axis and if f is real for real z, then

$$f(z) = \overline{f(\bar{z})}$$

pf: Thm 7.8 + Uniqueness Thm