Open mapping theorem

Recall

A map (between topological/metric spaces) is called open if it maps open sets to open sets.

Open Mapping Thm (Thm 7.1)

A nonconstant analytic map is open.

pf

Let $g : U \xrightarrow{\text{open } \mathbb{C}} \mathbb{C}$ be nonconstant and analytic. Let $\alpha \in U$, and let

$$f(z) = g(z) - g(\alpha) \quad \forall z \in U.$$

$$\Rightarrow \quad f(\alpha) = 0.$$

Lemma

$\exists\, C_r = \{\alpha + re^{i\theta} : \theta \in [0, 2\pi]\}$ s.t. $\overline{D(\alpha;r)} \subseteq U$ $f(z) \neq 0 \ \forall z \in C_r$

pf: If not, $\exists N\ \exists \theta_n \in [0,2\pi]$ s.t. $f(\alpha + \frac{1}{n} e^{i\theta_n}) = 0 \ \forall n \geq N$, i.e. $f = 0$ in $\{\alpha + \frac{1}{n} e^{i\theta_n} : n \geq N\} \subseteq U$ has acc pt $\alpha \in U$

$\overset{\text{uniqueness thm}}{\Rightarrow}$ $f \equiv 0$ ($\rightarrow\leftarrow$) #

Let

$$\varepsilon = \frac{1}{2} \min_{z \in C_r} |f(z)| \quad > 0$$

Claim

$$D(0;\varepsilon) \subseteq f(D(\alpha;r))$$

pf

Let $\omega \in D(0;\varepsilon)$. For $z \in C_r$,

$$|f(z) - \omega| \geq |f(z)| - |\omega| \geq 2\varepsilon - \varepsilon = \varepsilon$$

and at α,

$$|f(\alpha) - \omega| = |-\omega| < \varepsilon \leq |f(z) - \omega| \quad \forall z \in C_r$$

$\Rightarrow$ $|f(z) - \omega|$ assumes its minimum inside C.

By Minimum Modulus Thm (Thm 6.14), $f(z) - \omega = 0$ for some $z \in D(\alpha;r)$ #

can be done for each $\alpha \in U$

$$D(0;\varepsilon) + g(\alpha) \subseteq f(D(\alpha;r)) + g(\alpha)$$

$$D(g(\alpha);\underbrace{\varepsilon}_{=\varepsilon_\alpha}) \subseteq g(D(\alpha;r)) \subseteq g(U)$$

$$\Rightarrow g(U) = \bigcup_{\alpha \in U} D(g(\alpha);\varepsilon_\alpha) \text{ is open } \#$$

Functions on unit disc

Schwarz Lemma (Thm 7.2)

Suppose that (a) f is analytic in the unit disc $D = D(0;1)$, (b) $|f(z)| \leq 1 \ \forall z \in D$ and (c) $f(0) = 0$. Then

(i) $|f(z)| \leq |z| \quad \forall z \in D$

(ii) $|f'(0)| \leq 1$

with equality holds in (i) or (ii) at some point iff $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$

pf

Let

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & 0 < |z| < 1 \\ f'(0) & z = 0 \end{cases}$$

which is analytic by Prop 6.7.

By (b), $|g(z)| \leq \frac{1}{|z|} \longrightarrow 1$ as $|z| \to 1$.

Maximum Modulus Thm $\Rightarrow$ $|g(z)| \leq \max_{\theta} |g(re^{i\theta})| \leq \frac{1}{r} \to 1$ as $r \to 1$ $\forall |z| \leq r$ $\Rightarrow$ $|g(z)| \leq 1 \quad \forall z \in D$

$$\Rightarrow \begin{cases} |f(z)| \leq |z| \\ |f'(0)| \leq 1 \end{cases}$$

Furthermore, if "=" holds at some point z_0, then $|g(z_0)| = 1$ at $z_0 \in D$

By Maximum Modulus Thm, $g(z_0) = \text{constant} = e^{i\theta}$ for some $\theta \in \mathbb{R}$. $\Rightarrow f(z) = e^{i\theta} z$ #

Prop (p.95)

Suppose $|\alpha|<1$ and

$$B_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha} z}.$$

Note : $B_\alpha(\alpha)=0$

Then $B_\alpha : D=D(0;1) \longrightarrow D$ is analytic

pf

Note $1-\bar\alpha z = 0 \Leftrightarrow z = \frac{1}{\bar\alpha}$ and $\left|\frac{1}{\bar\alpha}\right| = \frac{1}{|\alpha|} > 1 \Rightarrow B_\alpha$ is analytic in D.

If $|z|=1$, then

$$|B_\alpha(z)|^2 = B_\alpha(z)\overline{B_\alpha(z)} = \frac{z-\alpha}{1-\bar\alpha z}\cdot\frac{\bar z-\bar\alpha}{1-\alpha\bar z} = \frac{z\bar z - z\bar\alpha - \alpha\bar z + \alpha\bar\alpha}{1-\bar\alpha z - \alpha\bar z + |\alpha|^2|z|^2}$$

$$= \frac{1+|\alpha|^2-(z\bar\alpha+\bar z\alpha)}{1+|\alpha|^2-(\bar\alpha z+\alpha\bar z)} = 1$$

By Maximum Modulus Thm, $|B_\alpha(z)| < 1 \quad \forall |z|<1$. #

Example (p.95-96)

① Suppose f is analytic and bounded by 1 in the unit disc and $f(\frac12)=0$.

Show that $|f(\frac34)| \le \frac25$ and "=" is achieved by some analytic function.

pf

Let

$$g(z) := \begin{cases} f(z)/B_{\frac12}(z) = \overset{|\cdot|\le 1}{f(z)} \Big/ \overset{|\cdot|=1 \text{ when } |z|=1}{\left(\frac{z-\frac12}{1-\frac12 z}\right)} & z\ne\frac12 \\ \frac34 f'(\frac12) = \lim_{z\to\frac12} f(z)/B_{\frac12}(z) & z=\frac12 \end{cases}$$

$\Rightarrow g(z)$ is analytic and $\lim_{|z|\to1}|g(z)| \le 1 \overset{\text{Max Mod Thm}}{\Longrightarrow} |g(z)|\le 1 \quad \forall |z|<1$

$\Rightarrow |f(z)| \le |B_{\frac12}(z)| = \left|\frac{z-\frac12}{1-\frac12 z}\right|$

In particular, $|f(\frac34)| \le |B_{\frac12}(\frac34)| = \frac25$

Note that "=" is achieved by $f(z)=B_{\frac12}(z)$ #

② Suppose $f: D=D(0;1)\to D$ is analytic s.t.

$$|f'(\tfrac13)| = \max\{|h'(\tfrac13)| : h: D\to D \text{ is analytic}\}$$

Show that $f(\frac13)=0$

pf

Suppose $f(\frac13)\ne 0$ and consider $g(z) = \dfrac{f(z)-f(\frac13)}{1-\overline{f(\frac13)}f(z)} = B_{f(\frac13)}(f(z))$

We have $g: D=D(0;1) \xrightarrow{f} D \xrightarrow{B_\alpha,\ \alpha=f(\frac13)} D$

In particular, $|g(z)|<1 \quad \forall |z|<1$.

Note that

$$|g'(\tfrac13)| = \left|\frac{f'(\frac13)\cancel{(1-\overline{f(\frac13)}f(\frac13))} - \overbrace{(f(\tfrac13)-f(\tfrac13))}^{=0}(-\overline{f(\frac13)}f'(\frac13))}{(1-\overline{f(\frac13)}f(\frac13))^{\cancel{2}}}\right| = \left|\frac{f'(\frac13)}{1-|f(\frac13)|^2}\right| \overset{\text{max by assumption}}{>} |f'(\tfrac13)| \ (\rightarrow\leftarrow)$$ #

$|f(\frac13)|^2 \ne 0$ by assumption

exer (exer 10, 11 in Ch7)

max $|f'(\frac13)|$ is assumed by $B_{\frac13}(z)$.

Converse of retangle thm

Recall (Rectangle Thm i.e. Thm 6.1)

Suppose f is analytic in $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $R\subseteq U$ is a rectangle. Then

$$\int_\Gamma f(z)\,dz = 0$$

where $\Gamma = \partial R$

Morera Thm (Thm 7.4)

Let f be a continuous function on an open set $U\subseteq\mathbb{C}$. If

$$\int_\Gamma f(z)\,dz = 0$$

whenever Γ is the boundary of a closed rectangle in U, then f is analytic in U.