☆ Thm 5.12 (Fundamental Theorem of Algebra)

Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$. (i.e. $\mathbb{C}$ is algebraically closed)

pf

Suppose $p(z)$ is a nonconstant polynomial s.t. $p(z) \neq 0$ $\forall z \in \mathbb{C}$

$\Rightarrow$ $f(z) := \frac{1}{p(z)}$ is an entire function

By Lemma, $\exists R > 0$ s.t. $|p(z)| > 1$ $\forall |z| > R$

By continuity of $f(z)$, $\exists M_0$ s.t. $|f(z)| \leq M_0$ $\forall |z| \leq R$

$\Rightarrow$ $|f(z)| \leq 1 + M_0$ $\forall z \in \mathbb{C}$

$\overset{\text{Liouville Thm}}{\Rightarrow}$ $f(z)$ is constant $\Rightarrow$ $p(z)$ is a constant. $(\rightarrow\leftarrow)$ #

Remark

① Recall that given polynomials $f(z), g(z) \in \mathbb{C}[z]$, $\exists!$ $q(z)$, $r(z)$ s.t. (division of poly)

- $f(z) = q(z) \cdot g(z) + r(z)$
- $r(z) = 0$ or $\deg(r) < \deg(g)$.

We say $g$ is a factor of $f$, denoted $g \mid f$, if $r = 0$

② $z - \alpha$ is a factor of a poly $f(z)$ $\Leftrightarrow$ $f(\alpha) = 0$

③ $\alpha$ is called a zero of multiplicity $k$ of $f(z)$ if $(z-\alpha)^k \mid f(z)$ and $(z-\alpha)^{k+1} \nmid f(z)$.

$\Leftrightarrow$ $f(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0$, $f^{(k)}(\alpha) \neq 0$

④ Suppose

$f(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_n \neq 0$

By Thm 5.12, $\exists z_1$ s.t. $f(z_1) = 0$ $\overset{②}{\Rightarrow}$ $f(z) = (z - z_1) q(z)$ for poly $q(z)$ of deg $n-1$

Apply Thm 5.12 to $q_1(z)$: $f(z) = (z-z_1) q_1(z) = (z-z_1)(z-z_2) q_2(z) = \cdots = (z-z_1)\cdots(z-z_n) \cdot \underset{=a_n}{C}$ $a_n$ by comparing coefficients

So $f(z) = a_n (z-z_1)\cdots(z-z_n)$

$= a_n (z-\alpha_1)^{m_1} \cdots (z-\alpha_k)^{m_k}$ ← multiplicity ← combine same $z_j$.

Thus,

(i) a polynomial of deg $n \geq 1$ has $n$ zeros in $\mathbb{C}$ counting multiplicities

(ii) by comparing coeff, one can get relations between zeros and coeff such as $\sum_{k=1}^{n} z_k = -\frac{a_{n-1}}{a_n}$

⑤ Recall that people extend $\mathbb{R}$ to $\mathbb{C}$ because $x^2 + 1 = 0$ has no zero in $\mathbb{R}$.

Thm 5.12 $\Rightarrow$ one DONOT need a further extension of $\mathbb{C}$ when solving polynomial equations.

exer: Thm 5.14

# Ch6-7 Further properties of analytic functions

## Uniqueness of analytic functions

Recall

- (Thm 2.12) Let $f(z)=\sum_{n=0}^{\infty} C_n z^n$. If $\exists$ nonzero seq $\{z_k\}$ s.t. ① $\lim_{k\to\infty} z_k = 0$ ② $f(z_k)=0\ \forall k$, then $f\equiv 0$
- (Thm 6.5) If $f$ is analytic in $D(\alpha; r)$, then $\exists\ C_n$ s.t. $f(z)=\sum_{n=0}^{\infty} C_n(z-\alpha)^n \quad \forall z\in D(\alpha; r)$
- (Advanced Calculus) Let $D$ be a connected set. If ① $D=A\cup B$, ② $A, B$ are open in $D$, ③ $A\cap B=\phi$, then $A=\phi$ or $D$, $B=D$ or $\phi$

Thm 6.9 (Uniqueness Thm)

Suppose that $f$ is analytic in a region (i.e. open connected set) $D$ and that $f(z_n)=0$ where $\{z_n\}$ is a seq of distinct points and $z_n\longrightarrow z_0\in D$. Then $f\equiv 0$ in $D$.

pf

Let

$$A=\{\, z\in D:\ \exists \text{ distinct } \omega_n\in D \text{ s.t. } f(\omega_n)=0 \text{ and } \omega_n\to z\,\}$$

(⇒ $f(z)=0$ by continuity of $f$)

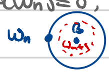
$$B = D\setminus A$$

Then ① $A\cup B=D$, $A\cap B=\phi$

② $\forall\,\alpha\in A$, $\exists\,\varepsilon>0$, $\exists\, C_n\in\mathbb{C}$ s.t. $f(z)=\sum_{n=0}^{\infty} C_n(z-\alpha)^n \quad \forall z\in D(\alpha;\varepsilon)\subseteq D$.

By the uniqueness thm of power series (Thm 2.12), $f(\omega_n)=0$, $\omega_n\neq\omega_m$, $\omega_n\to\alpha \Rightarrow C_n=0\ \forall n$ i.e. $f\equiv 0$ in $D(\alpha;\varepsilon)$

$\Rightarrow D(\alpha;\varepsilon)\subseteq A \Rightarrow A$ is open




③ Given $\beta\in B$, claim $\exists\,\varepsilon>0$ s.t. $D(\beta;\varepsilon)\subseteq B$. If not, $\exists\ \omega_n\in A$ s.t. $\omega_{n+1}\in D(\beta;\tfrac{1}{2}|\omega_n-\beta|)\cap A$

$\Rightarrow \omega_n\neq\omega_m\ \forall n\neq m$, $\in D$ and $\omega_n\to\beta \Rightarrow \beta\in A$ (→←)

So $\exists\,\varepsilon>0$ s.t. $D(\beta;\varepsilon)\subseteq B \Rightarrow B$ is open

So assumption $\Rightarrow A\neq\phi \xRightarrow{D \text{ is connected}} A=D \Rightarrow f\equiv 0$ in $D$ #

Cor 6.10

If two functions $f$ and $g$, analytic in a region $D$, agree at a set of points with an accumulation point in $D$, then $f\equiv g$ in $D$.

pf Consider $f-g$. #

Remark

$\sin(1/z)=0$ on the set $\{\frac{1}{n\pi}: n=\pm1,\pm2,\cdots\}$ which has an acc point $0$ but $\sin(1/z)\not\equiv 0$ !!

This can happen because $\sin(1/z)$ is NOT analytic at $0$. ⇒ The hypothesis of Thm 6.9 is NOT satisfied.

Applications of uniqueness thm:

① Prove functional equations such as

$$e^{z_1+z_2}=e^{z_1}e^{z_2}$$

pf: Given any fixed $z_2\in\mathbb{C}$, $e^{z_1+z_2}$ and $e^{z_1}e^{z_2}$ are entire and $e^{x+z_2}=e^{x}e^{z_2}\quad\forall x\in\mathbb{R}$ (has acc point in $\mathbb{C}$)

$\Rightarrow e^{z_1+z_2}=e^{z_1}e^{z_2}\quad \forall z_1\in\mathbb{C}$ #

② Thm 6.11

If $f$ is entire and if $f(z)\to\infty$ as $z\to\infty$, then $f$ is a polynomial

pf

By assumption, $\exists\, M>0$ s.t. $|z|>M \Rightarrow |f(z)|>1$

Lemma

$f$ has a finite number of zeros

pf: If not, $\exists\ |z_n|\le M$, $z_n\neq z_m\ \forall n\neq m$, $n=1,2,\cdots$ s.t. $f(z_n)=0 \xRightarrow{\overline{D(0;M)} \text{ is compact}} \{z_n\}$ has an acc point $\Rightarrow f\equiv 0$ in $\mathbb{C}$ (→←) #

Let $\alpha_1,\cdots,\alpha_N$ be the zeros of $f$ ($\Rightarrow |\alpha_j|\le M$), and let $k_j$ be the number s.t.

$$f(\alpha_j)=f^{(1)}(\alpha_j)=\cdots=f^{(k_j-1)}(\alpha_j)=0,\quad f^{(k_j)}(\alpha_j)\neq 0$$

pf

Lemma

$k_j < \infty$

pf If not, then $f(\alpha_j) = f^{(1)}(\alpha_j) = \cdots = 0 \Rightarrow$ the power series of $f$ at $\alpha_j$ is $0$ $\Rightarrow f \equiv 0$ $\Leftrightarrow\Leftarrow$

Let

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-\alpha_1)^{k_1}(z-\alpha_2)^{k_2}\cdots(z-\alpha_N)^{k_N}} & \text{if } z \neq \alpha_1, \cdots, \alpha_n \\ \lim\limits_{z\to\alpha_j} \dfrac{f(z)}{(z-\alpha_1)^{k_1}\cdots(z-\alpha_N)^{k_N}} & \text{if } z = \alpha_j \end{cases}$$

By applying Cor 5.9 repeatedly, we conclude $g(z)$ is entire.

Also note that $g(z) \neq 0 \ \forall z \in \mathbb{C}$ ← exer

$\Rightarrow h(z) := 1/g(z)$ is also entire ($\Rightarrow$ Continuous)

Since $h$ is continuous, $\exists B > 0$ s.t. $|h(z)| \leq B \ \forall \ |z| \leq M$.

For $|z| > M$, $|f(z)| > 1$

$$\Rightarrow |h(z)| = |z-\alpha_1|^{k_1}\cdots|z-\alpha_N|^{k_N} / |f(z)| < |z-\alpha_1|^{k_1}\cdots|z-\alpha_N|^{k_N}$$
$$\leq (|z|+|\alpha_1|)^{k_1}\cdots(|z|+|\alpha_N|)^{k_N} \qquad (\text{Note: } |\alpha_j| \leq M < |z|)$$
$$\leq (|z|+|z|)^{k_1}\cdots(|z|+|z|)^{k_N} = 2^{k_1+\cdots+k_N}|z|^{k_1+\cdots+k_N}$$

Let $A = 2^{k_1+\cdots+k_N}$ and $d = k_1+\cdots+k_N$. Then

$$|h(z)| \leq A\cdot|z|^d + B \qquad \forall z \in \mathbb{C}$$

Liouville Thm (Thm 5.11) $\Rightarrow$ $h$ is a polynomial

Since $h(z) \neq 0 \ \forall z \in \mathbb{C}$, by the Fundamental Thm of Alg, $h$ is a constant $c^{\neq 0}$.

$\Rightarrow g(z) \equiv \frac{1}{c} \Rightarrow f(z) = \frac{1}{c}(z-\alpha_1)^{k_1}\cdots(z-\alpha_N)^{k_N}$ is a polynomial. #

## Max/min modulus thm

### Mean Value Thm (Thm 6.12)

If $f$ is analytic in $D$ and $\alpha \in D$, then

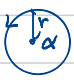
$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\,d\theta$$

when $\overline{D(\alpha;r)} \subset D$

pf

By Cauchy Integral Formula (Thm 6.4)

$$f(\alpha) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-\alpha}\,dz$$

where $C_r$ is the circle . By the parameterization $z = \alpha + re^{i\theta}$, we have

$$f(\alpha) = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(\alpha+re^{i\theta})}{re^{i\theta}}\cdot \cancel{i}re^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha+re^{i\theta})\,d\theta$$ #

### Maximum Modulus Thm (Thm 6.13)

Suppose $f$ is a nonconstant analytic function in a region $D$. Then $\forall z \in D \ \forall \delta > 0$, $\exists\, \omega \in D(z;\delta) \cap D$ s.t. $|f(\omega)| > |f(z)|$

pf

Suppose $\exists z_0, \exists \delta > 0$ s.t. $|f(\omega)| \leq |f(z_0)| \ \forall \omega \in D(z_0;\delta) \cap D$. (open)

Recall (Lemma 4.9) $\left|\int_a^b G(t)dt\right| \leq \int_a^b |G(t)|dt$

Let $r > 0$ be any number s.t. $\overline{D(z_0;r)} \subseteq D(z_0;\delta) \cap D$.

$\Rightarrow f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$

$\Rightarrow |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|f(z_0)|\,d\theta = |f(z_0)|$

$$\Rightarrow \begin{cases} \frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta = |f(z_0)| \\ |f(z_0+re^{i\theta})| \leq |f(z_0)| \ \forall \theta \in [0,2\pi] \end{cases} \xRightarrow{\text{continuity of } f} |f(z_0)| = |f(z_0+re^{i\theta})| \ \forall \theta \in [0, 2\pi]$$

pf

Thus, we have $|f(w)| = |f(z_0)|$ $\forall\, w \in D(z_0; r)$

Recall from Prop 3.7 that if $g$ is analytic in a region $U$ and $|g|$ is constant in $U$, then $g$ is constant.

So $f$ is constant in $D(z_0; r)$ $\Rightarrow$ $f$ is constant in $D$ $(\rightarrow\leftarrow)$ #

Cor

Suppose $f$ is analytic in a bounded region $D$ and continuous on $\bar{D}$, then $|f|$ has a maximum in the boundary $\partial D$ of $D$

open connected

pf

Since $|f|$ is continuous on $\bar{D}$ and $\bar{D}$ is compact, $|f|$ has a maximum at a point $p \in \bar{D}$.

By Max Modulus Thm, $p \notin D$ $\Rightarrow$ $p \in \partial D = \bar{D} \setminus D$. #

Minimum Modulus Theorem (Thm 6.14)

If $f$ is a non-constant analytic function in a region $D$, and $\exists\, \delta > 0$ s.t.

$$|f(z)| \leq |f(w)| \qquad \forall\, w \in D(z; \delta) \cap D,$$

then $f(z) = 0$

pf

Suppose $f(z) \neq 0$, Then $g = \frac{1}{f}$ is analytic in $D(z;\delta) \cap D$ and

$$|g(z)| = \frac{1}{|f(z)|} \geq \frac{1}{|f(w)|} = |g(w)| \qquad \forall\, w \in D(z;\delta) \cap D$$

$\overset{\text{Max Modulus Thm}}{\Rightarrow}$ $g = \text{constant}$ $(\rightarrow\leftarrow)$

So $f(z) = 0$ #

Application: a Liouville-type theorem

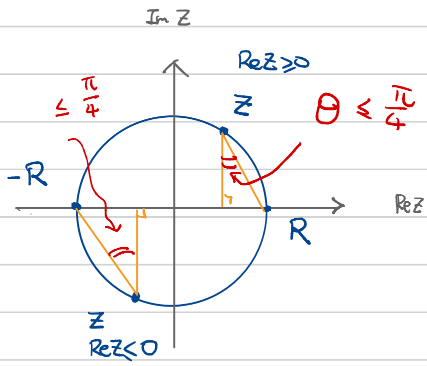
Prop 7.3

If $f$ is an entire function satisfying

$$|f(z)| \leq \frac{1}{|\operatorname{Im} z|} \qquad \forall\, z \in \mathbb{C},$$

then $f \equiv 0$.

pf




Let

$$g(z) = (z^2 - R^2) f(z)$$

Given any $z$ s.t. $|z| = R$, $\operatorname{Re} z \geq 0$,

$$|(z-R) f(z)| \leq |z-R| / |\operatorname{Im} z| = \sec\theta$$

for some $\theta \in [0, \frac{\pi}{4}]$

$\Rightarrow$ $|(z-R) f(z)| \leq \sqrt{2}$

Similarly, if $|z| = R$, $\operatorname{Re} z \leq 0$, then

$$|(z+R) f(z)| \leq \sqrt{2}$$

Thus, for any $z$, $|z| = R$,

$$|g(z)| = |z+R||z-R||f(z)| \leq \left\{ \begin{array}{ll} |z+R| \cdot \sqrt{2} \leq (|z|+R)\sqrt{2} = 2\sqrt{2}\, R \leq 3R & \text{if } \operatorname{Re} z \geq 0 \\ |z-R| \cdot \sqrt{2} \leq 3R & \text{if } \operatorname{Re} z \leq 0 \end{array} \right\} \leq 3R$$

By Maximum Modulus Thm,

$$|g(z)| = |z^2 - R^2||f(z)| \leq 3R \qquad \forall\, |z| < R$$

$\Rightarrow$ $|f(z)| \leq \frac{3R}{|z^2 - R^2|}$ $\forall\, R > |z|$ ← for any fixed $z$, any $R > |z|$

$\Rightarrow$ $|f(z)| = \lim_{R\to\infty} |f(z)| \leq \overline{\lim_{R\to\infty}} \frac{3R}{|z^2 - R^2|} = 0$ $\forall\, z$

$\Rightarrow$ $f \equiv 0$ #