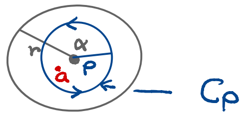


Thm 5.3 (Cauchy Integral Formula, also see Thm 6.4)

Suppose f is analytic in $D(\alpha; r)$ for some $r \in (0, \infty]$. Suppose $\rho \in (0, r)$ and $|a - \alpha| < \rho$.

Then
$$f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-a}\, dz$$
where C_ρ is the circle $\alpha + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$.

pf

By Closed Curve Thm (Thm 4.16, Cor 5.2, Thm 6.3),
$$\int_{C_\rho} \frac{f(z) - f(a)}{z-a}\, dz = 0$$
$$\Rightarrow \int_{C_\rho} \frac{f(z)}{z-a}\, dz = f(a) \int_{C_\rho} \frac{1}{z-a}\, dz \overset{\text{Lemma 5.4}}{=} 2\pi i \cdot f(a)$$
$$\Rightarrow f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-a}\, dz$$
#

Taylor expansion of an analytic function

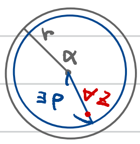
Thm 5.5 (Also see Thm 6.5, Thm 6.6)

If f is analytic in $D(\alpha; r)$, $r \in (0, \infty]$, then $f^{(n)}(\alpha)$ exists for $n = 1, 2, \cdots$, and
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n$$
$\forall\, z \in D(\alpha; r)$.

pf

Let $\rho \in (0, r)$ and C_ρ be the circle $\alpha + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$. By Cauchy Integral Formula, $\forall z \in D(\alpha; \rho)$,
$$f(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{w-z}\, dw$$
Recall, in the proof of Lemm 5.4, we proved that
$$\sum_{n=0}^{\infty} \frac{(z-\alpha)^n}{(w-\alpha)^{n+1}} \longrightarrow \frac{1}{w-\alpha} \cdot \frac{1}{1 - \frac{z-\alpha}{w-\alpha}} = \frac{1}{w-z}$$
(as functions in w) uniformly on C_ρ

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_{C_\rho} \sum_{n=0}^{\infty} f(w) \frac{(z-\alpha)^n}{(w-\alpha)^{n+1}}\, dw$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{(w-\alpha)^{n+1}}\, dw \right) \cdot (z-\alpha)^n \qquad \forall\, z \in D(\alpha; \rho)$$
Note that the numbers
$$c_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{(w-\alpha)^{n+1}}\, dw$$
are independent of z. So, by the uniqueness of power series (Cor 2.11, Thm 2.12),
$$c_n = \frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)}{(w-\alpha)^{n+1}}\, dw$$



Since ρ is artibrary in $(0, r)$, we can conclude
$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z-\alpha)^n \qquad \forall\, z \in D(\alpha; r)$$
#

Cor 5.6 (Also see Thm 6.8)

Any analytic function is infinitely differentiable

pf

If f is analytic at α, (by def) then $\exists r > 0$ s.t. f is analytic in $D(\alpha; r)$ $\overset{\text{Thm 5.5}}{\Rightarrow}$ f is ∞ differentiable at α
#

Example

① Suppose f is entire and $f(\frac{1}{n})=0\ \forall n \Rightarrow f\equiv 0$ (Thm 5.5 + Thm 2.12 (Uniqueness of power series))

② $f(z):=\begin{cases}\exp(\frac{-1}{(|z|-1)^2}) & |z|>1\\ 0 & |z|\le 0\end{cases}$ is smooth, $f(\frac{1}{n})=0\ \forall n$

Remark (an idea about how good a function is)

"CR eq" — difference is still big; difference is not so big

Bad — arbitrary function — Continuous — real-valued C^∞ — (complex) analytic — (complex) poly → Good

Continuous, C^∞: CANNOT be determined by its values on a non-dense subset

analytic: determined by its values on a set with an acc pt (Cor 2.14, Cor 6.10) 2/14

poly: determined by its values on a finite set

Prop 5.8 (Also see Prop 6.7)

If f is analytic in an open set $U\subseteq\mathbb{C}$ and $a\in U$, then

$$g(z):=\begin{cases}\frac{f(z)-f(a)}{z-a} & \text{if } z\in U-\{a\}\\ f'(a) & \text{if } z=a\end{cases}$$

is also analytic in U.

pf

g is clearly analytic in $U\setminus\{a\}$. It suffices to show g is analytic at a: let $r>0$ s.t. $D(a;r)\subseteq U$.

$\Rightarrow\quad g(z)=\frac{1}{z-a}\left(\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n-f(a)\right)=\sum_{n=1}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^{n-1}\quad\forall z\in D(a;r)\setminus\{a\}$

Note that $g(a)=f'(a)=\sum_{n=1}^{\infty}\frac{1}{n!}f^{(n)}(a)(a-a)^{n-1}$

$\Rightarrow\quad g(z)=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}f^{(n+1)}(a)(z-a)^n\quad\forall z\in D(a;r)\quad\Rightarrow\ g$ is analytic at a #

Cor 5.9

Suppose f is analytic in an open set $U\subseteq\mathbb{C}$, $a_1,\cdots,a_N\in U$ (distinct), and $f(a_k)=0$, $k=1,\cdots,N$. Then $\lim_{z\to a_k}\frac{f(z)}{(z-a_1)\cdots(z-a_N)}$ exists for $k=1,\cdots,N$.

and the function

$$g(z):=\begin{cases}\frac{f(z)}{(z-a_1)(z-a_2)\cdots(z-a_N)} & \forall z\in U\setminus\{a_1,\cdots,a_N\}\\ \lim_{z\to a_k}\frac{f(z)}{(z-a_1)\cdots(z-a_N)} & z=a_k,\ k=1,\cdots,N\end{cases}$$

is analytic in U.

pf (induction)

Let $f_0(z)=f(z)$ and let

$$f_1(z):=\begin{cases}\frac{f_0(z)-f_0(a_1)}{z-a_1}=\frac{f_0(z)}{z-a_1}, & z\in U\setminus\{a_1\}\\ f_0'(a_1)=\lim_{z\to a_1}\frac{f_0(z)}{z-a_1} & z=a_1\end{cases}$$

Prop 5.8 $\Rightarrow f_1(z)$ is analytic in U. And $f_1(a_k)=0$, $k=2,\cdots N$.

Inductively, let

$$f_k(z):=\begin{cases}\frac{f_{k-1}(z)-f_{k-1}(a_k)}{z-a_k}=\frac{f_{k-1}(z)}{z-a_k}, & z\in U\setminus\{a_k\}\\ f_{k-1}'(a_k)=\lim_{z\to a_k}\frac{f_{k-1}(z)}{z-a_k}, & z=a_k\end{cases}$$

Then Prop 5.8 $\Rightarrow f_k(z)$ is analytic in U. And $f_k(a_\ell)=0$, $\ell=k+1,\cdots,N$.

So we have $g(z)=f_N(z)$ is analytic in U. #

Remark (§6.2, p.81)

Let $U\subseteq\mathbb{C}$ be open. If $f:U\to\mathbb{C}$ is analytic and $D(a;r)\subseteq U$, then $\exists\, c_n\in\mathbb{C}$ s.t.

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n\qquad\forall z\in D(a;r)$$

Note that "$\forall z\in D(a;r)$" CANNOT be replaced by "$\forall z\in U$"

Examples (p. 81-82)

(i) $f(z)=\frac{1}{z-1}$ is analytic in $\mathbb{C}\setminus\{1\}$ and, in particular, analytic at 2. Furthermore,

$$f(z)=\frac{1}{1+z-2}=\sum_{n=0}^{\infty}(-1)^n(z-2)^n\qquad\forall z\in D(2;1)$$

(exer: $\nexists\sum_{n=0}^{\infty}c_n(z-a)^n$ s.t. $\frac{1}{z-1}=\sum_{n=0}^{\infty}c_n(z-a)^n\quad\forall z\in\mathbb{C}\setminus\{1\}$)

But this equality is NOT true for $|z-2|>1$.

Also note that $f(z)=\sum_{n=0}^{\infty}-z^n\quad\forall\ |z|<1$

Different expansions near different points!!

Remark {entire functions} = {power series with ∞ radii of conv}

{analytic functions} = {functions which can be expressed by power series locally}

(ii) Find a power series representation for $\frac{1}{z^2}$ near 3:

$$\frac{1}{z^2}=\left(\frac{1}{3+(z-3)}\right)^2=\frac{1}{9}\left(\frac{1}{1+(z-3)/3}\right)^2=\frac{1}{9}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(z-3)^n\right)^2=\sum_{n=0}^{\infty}\frac{1}{9}\left(\sum_{k=0}^{n}\frac{(-1)^k}{3^k}\cdot\frac{(-1)^{n-k}}{3^{n-k}}\right)(z-3)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{9}\frac{n+1}{3^n}(z-3)^n$$

converges when $|z-3|<$ radius of convergence of $\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(z-3)^n$

for $|z-3|<1\Big/\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{3^n}\right|}=3$

Fundamental Thm of Algebra

☆ Thm 5.10 (Liouville Thm)

A bounded entire function is constant

pf

Let $a, b \in \mathbb{C}$, and M be an upper bound of $|f(z)|$, $z\in\mathbb{C}$. Then, by Cauchy Integral Formula, $\forall R > \max\{|a|, |b|\}$

$$|f(b) - f(a)| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-b}\,dz - \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-a}\,dz \right|$$

$C_R : Re^{i\theta},\ 0 \le \theta \le 2\pi$

indep of R

$$= \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z)\cdot(b-a)}{(z-a)(z-b)}\,dz \right|$$

$$\le \underbrace{2\pi R}_{\text{length}(C_R)} \cdot \frac{1}{2\pi} \frac{M\cdot|b-a|}{(R-|a|)(R-|b|)}$$

Note $|z-a| \ge |z| - |a| = R - |a| \quad \forall z \in C_R$

$$\longrightarrow 0 \quad \text{as } R \to \infty$$

$\Rightarrow f(a) = f(b)$ #

Thm 5.11 (Extended Liouville Thm)

If f is entire and if, for some $k \ge 0$, there exist constants $A, B \ge 0$ s.t.

$$|f(z)| \le A + B|z|^k \qquad \forall z \in \mathbb{C},$$

then f is a polynomial of degree at most k.

pf (induction on k)

The case $k=0$ is the original Liouville Thm. Assume Thm 5.11 is true for $k-1$.

If $|f(z)| \le A + B|z|^k$ $\forall z \in \mathbb{C}$, then, by Prop 5.8, the function

$$g(z) = \begin{cases} \dfrac{f(z)-f(0)}{z}, & z \ne 0 \\ f'(0) \end{cases}$$

is entire, and

$$|g(z)| = \frac{|f(z)-f(0)|}{|z|} \le \frac{A + B|z|^k + |f(0)|}{|z|} \le A + |f(0)| + B|z|^{k-1} \qquad \forall |z| > 1$$

By continuity of $g(z)$, $\exists M$ s.t. $|g(z)| \le M$ $\forall |z| \le 1$

$\Rightarrow$ if we take $D = \max\{M, A + |f(0)|\}$, $E = B$, then

$$|g(z)| \le D + E|z|^{k-1} \qquad \forall z \in \mathbb{C}$$

$\overset{\text{induction hypothesis}}{\Rightarrow}$ $g(z)$ is a polynomial of deg at most $k-1$

$\Rightarrow f(z) = f(0) + g(z)\cdot z$ is a poly of deg at most k #

Lemma

If $p(z)$ is a nonconstant (i.e. $\deg p(z) \ge 1$) polynomial, then

$$p(z) \longrightarrow \infty \quad \text{as } z \to \infty.$$

That is, $\forall M > 0$, $\exists R > 0$ s.t.

$$|p(z)| > M \quad \forall |z| > R$$

pf (induction on $k = \deg p(z)$)

$k = \deg p(z) = 1$: $p(z) = az + b$, $a \ne 0$

$\forall M > 0$, $\exists R = \frac{M + |b|}{|a|} > 0$ s.t. $|p(z)| = |az+b| \ge |a||z| - |b| > |a|\cdot\frac{M+|b|}{|a|} - |b| = M$ $\quad \forall |z| > R$

$k-1 \Rightarrow k$: $p(z) = a_k z^k + \cdots + a_0$, $a_k \ne 0$

By induction hypothesis, $\exists R_0 > 0$ s.t. $|a_k z^{k-1} + \cdots + a_1| > 1$ $\quad \forall |z| > R_0$

$\Rightarrow \forall M > 0$, $\exists R = |a_0| + M + R_0 > 0$ s.t.

$$|p(z)| \ge |a_k z^{k-1} + \cdots + a_1||z| - |a_0| > 1\cdot(|a_0| + M + R_0) - |a_0| > M \qquad \forall |z| > R$$

This completes the induction. #

Thm 5.12 (Fundamental Theorem of Algebra)

Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$. (i.e. $\mathbb{C}$ is algebraically closed)

pf

Suppose $p(z)$ is a nonconstant polynomial s.t. $p(z) \neq 0 \quad \forall z \in \mathbb{C}$

$\Rightarrow f(z) := \frac{1}{p(z)}$ is an entire function

By Lemma, $\exists R > 0$ s.t. $|p(z)| > 1 \quad \forall |z| > R$

By continuity of $f(z)$, $\exists M_0$ s.t. $|f(z)| \leq M_0 \quad \forall |z| \leq R$

$\Rightarrow |f(z)| \leq 1 + M_0 \quad \forall z \in \mathbb{C}$

$\overset{\text{Liouville Thm}}{\Rightarrow}$ $f(z)$ is constant $\Rightarrow$ $p(z)$ is a constant. ($\rightarrow\leftarrow$) #

Remark

① Recall that given polynomials $f(z), g(z) \in \mathbb{C}[z]$, $\exists!\, g(z), r(z)$ s.t. (division of poly)

- $f(z) = g(z) \cdot g(z) + r(z)$
- $r(z) = 0$ or $\deg(r) < \deg(g)$.

We say g is a factor of f, denoted $g \mid f$, if $r = 0$

② $z - \alpha$ is a factor of a poly $f(z) \Leftrightarrow f(\alpha) = 0$

③ α is called a zero of multiplicity k of $f(z)$ if $(z-\alpha)^k \mid f(z)$ and $(z-\alpha)^{k+1} \nmid f(z)$.

$\Leftrightarrow f(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0, \quad f^{(k)}(\alpha) \neq 0$

④ Suppose

$f(z) = a_n z^n + \cdots + a_1 z + a_0, \qquad a_n \neq 0$

By Thm 5.12, $\exists z_1$ s.t. $f(z_1) = 0 \overset{②}{\Rightarrow} f(z) = (z - z_1)\, g(z)$ for poly $g(z)$ of deg $n-1$

Apply Thm 5.12 to $g_1(z)$: $f(z) = (z-z_1)\, g_1(z) = (z-z_1)(z-z_2)\, g_2(z) = \cdots = (z-z_1)\cdots(z-z_n) \cdot C$ $\quad$ $= a_n$ by comparing coefficients

So $\quad f(z) = a_n (z-z_1)\cdots(z-z_n)$

$\quad = a_n (z-\alpha_1)^{m_1} \cdots (z-\alpha_k)^{m_k}$ ← multiplicity ← combine same z_j.

Thus,

(i) a polynomial of deg $n \geq 1$ has n zeros in $\mathbb{C}$ counting multiplicities

(ii) by comparing coeff, one can get relations between zeros and coeff such as $\sum_{k=1}^{n} z_k = -\frac{a_{n-1}}{a_n}$

⑤ Recall that people extend $\mathbb{R}$ to $\mathbb{C}$ because $x^2 + 1 = 0$ has no zero in $\mathbb{R}$.

Thm 5.12 $\Rightarrow$ one DONOT need a further extension of $\mathbb{C}$ when solving polynomial equations.

exer: Thm 5.14