Thm5.1 (Rectangle Thm II, also see Thm6.1)

Suppose f is analytic on $U \underset{\text{open}}{\subseteq} \mathbb{C}$, and $R \subseteq U$. For $a \in U$, let

$$g(z) := \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \in U \setminus \{a\} \\ f'(a), & z = a. \end{cases}$$

Then $\int_\Gamma g(z)\,dz = 0$

pf

case1 $a \notin R$

Note that $\frac{1}{z-a}$ is analytic on $\mathbb{C} \setminus \{a\} \supseteq R$ $\Rightarrow g = \frac{f(z)-f(a)}{z-a}$ is also analytic on $U \setminus \{a\} \supseteq R$

$\Rightarrow$ by Thm4.14, $\int_\Gamma g(z)\,dz = 0$

case2 $a \in \Gamma = \partial R$

Note that g is continuous on R (compact) $\Rightarrow \exists M > 0$ s.t. $|g(z)| \le M \ \forall z \in R$

We divide R into 6 subrectangles:

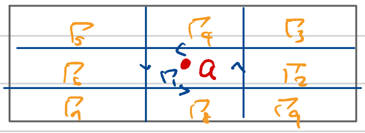
$$\Rightarrow \int_\Gamma g(z)\,dz = \sum_{j=1}^{6} \int_{\Gamma_j} g(z)\,dz \underset{\text{by case1}}{=} \int_{\Gamma_1} g(z)\,dz$$

Note that $\forall \varepsilon > 0$, $\exists$ division s.t. $\text{length}(\Gamma_1) < \varepsilon$

$\Rightarrow |\int_\Gamma g(z)\,dz| = |\int_{\Gamma_1} g(z)\,dz| \le M \cdot \varepsilon \quad \forall \varepsilon > 0 \quad \Rightarrow \int_\Gamma g(z)\,dz = 0$

case3 $a \in \text{int}(R)$

Similar as case2, we divide R:



$\forall \varepsilon > 0$, $\exists$ such division$_9$ s.t. $\text{length}(\Gamma_1) < \varepsilon$

$\Rightarrow |\int_\Gamma g(z)\,dz| = |\sum_{j=1}^{9} \int_{\Gamma_j} g(z)\,dz| \overset{\text{case1}}{=} |\int_{\Gamma_1} g(z)\,dz| \le M \cdot \varepsilon \quad \forall \varepsilon > 0 \quad \Rightarrow \int_\Gamma g(z)\,dz = 0$ #

Closed curve thm

Def4.13

A curve C given by $z(t)$, $a \le t \le b$, is called closed if $z(a) = z(b)$.

A closed curve C is called a simple closed curve if $\nexists$ other points coincide, i.e., $\underset{s \ne t}{z(s) = z(t)} \Rightarrow \{s,t\} = \{a,b\}$

e.g. ①  is closed, but not simple closed ②  is simple closed

Thm4.15 (Integral Thm, also see Cor5.2, Thm6.2)

Suppose $r \in (0, \infty]$. If $f : D(z_0; r) \overset{\subseteq \mathbb{C}}{\longrightarrow} \mathbb{C}$ is analytic, then $\exists$ analytic functions F, G on $D(z_0; r)$

s.t. $F'(z) = f(z)$ and $G'(z) = g(z) \quad \forall z \in D(z_0; r)$,

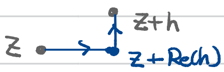
where $g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \in D(z_0; r) \setminus \{a\} \\ f'(a), & z = a \end{cases}$

Notation: $D(z_0; \infty) = \mathbb{C}$

pf

We will construct $F(z)$ by using Thm4.14. The construction of $G(z)$ is similar by applying Thm5.1.

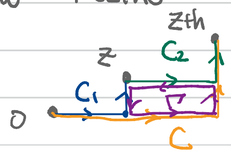
Notation:

$\int_z^{z+h} f(\zeta)\,d\zeta = \int_C f(\zeta)\,d\zeta$, $C =$ 

Special case: $z_0 = 0$

Let $F(z) := \int_0^z f(\zeta)\,d\zeta$

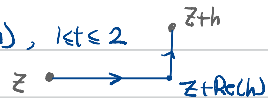
Note that $F(z+h) = F(z) + \int_z^{z+h} f(\zeta)\,d\zeta$ because



$$F(z+h) - (F(z) + \int_z^{z+h} f(\zeta)\,d\zeta) = \int_C f(\zeta)\,d\zeta - (\int_{C_1} f(\zeta)\,d\zeta + \int_{C_2} f(\zeta)\,d\zeta)$$
$$= \int_\Gamma f(\zeta)\,d\zeta = 0 \quad \text{(by Rectangle Thm)}$$

pf

$\gamma(t)=\begin{cases} z+t\,Re(h), & 0\le t\le 1 \\ z+Re(h)+i(t-1)Im(h), & 1\le t\le 2 \end{cases}$



Also note that $\int_z^{z+h} 1\, d\zeta = \int_0^1 Re(h)\, dt + \int_1^2 i\, Im(h)\, dt = h$

$\Rightarrow \dfrac{F(z+h)-F(z)}{h} - f(z) = \dfrac{1}{h}\left(\int_z^{z+h} f(\zeta)d\zeta - \int_z^{z+h} f(z)\, d\zeta\right) = \dfrac{1}{h}\int_z^{z+h} f(\zeta)-f(z)\, d\zeta$

Finally, since f is continuous at z, $\forall \varepsilon>0\ \exists \delta$ s.t. $|f(\zeta)-f(z)|<\varepsilon\ \ \forall\ |\zeta-z|<\delta$

$\Rightarrow \forall\ |h|<\delta$,

$\left|\dfrac{F(z+h)-F(z)}{h} - f(z)\right| = \dfrac{1}{|h|}\left|\int_z^{z+h} f(\zeta)-f(z)\, d\zeta\right| \le \dfrac{1}{|h|}\cdot\varepsilon\cdot(|Re(h)|+|Im(h)|) \le \dfrac{1}{|h|}\varepsilon\cdot 2|h| = 2\varepsilon$

$\Rightarrow F'(z) = f(z)$

general case: any z_0, $f: D(z_0;r)\to\mathbb{C}$

Let $h(z)=f(z+z_0)$. $\Rightarrow h: D(0;r)\to\mathbb{C}$ is analytic $\overset{\text{special case}}{\Rightarrow} \exists\ H: D(0;r)\to\mathbb{C}$ s.t. $H'(z)=h(z)$

Let $F(z)=H(z-z_0) \Rightarrow F'(z)=H'(z-z_0)=h(z-z_0)=f(z-z_0+z_0)=f(z)$ □

Thm 4.16 (Closed Curve Thm, also see Cor 5.2, Thm 6.3)

Suppose $r\in(0,\infty]$. If $f: D(z_0;r)\to\mathbb{C}$ is analytic, then for any closed piecewise C^1 curve C in $D(z_0;r)$,

(i) $\int_C f(z)\, dz = 0$

(ii) $\int_C g(z)\, dz = 0$

where $g(z)=\begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z\in D(z_0;r)\setminus\{a\} \\ f'(a), & z=a \end{cases}$

pf

By Integral Thm, $\exists$ analytic $F: D(z_0;r)\to\mathbb{C}$ s.t. $F'(z)=f(z)$ (resp. $F'(z)=g(z)$) in $D(z_0;r)$

$\Rightarrow$ by Prop 4.12, (resp. $\int_C g(z)dz$)

$\because C$ is closed

$$\int_C f(z)\, dz = \int_C F'(z)\, dz = F(z(b)) - F(z(a)) = 0$$ ✱

Ch5 Cauchy integral formula

Recall (from Advanced Calculus)

① (p.15) Let $D \subseteq \mathbb{C}$, and f, $f_n : D \to \mathbb{C}$, $n = 1, 2, \cdots$, be a seq of functions.

We say $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly in D if $\forall \varepsilon > 0$ $\exists N$ (indep of z) s.t.

$|f_n(z) - f(z)| < \varepsilon$ $\quad \forall n > N \quad \forall z \in D$

We say $\sum_{n=1}^{\infty} f_n$ converges uniformly in D if $\{\sum_{k=1}^{n} f_k\}_{n=1}^{\infty}$ converges uniformly in D

② If $f_n \to f$ uniformly and f_n are continuous in D, then f is continuous in D.

③ Weierstrass M-test (Thm 1.9, p.15)

Suppose $|f_n(z)| \leq \underset{\text{constants}}{M_n}$ $\forall z \in D$, $n = 1, 2, \cdots$. If $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n$ converge uniformly in D

Cor

If $\sum_{n=0}^{\infty} C_n (z - z_0)^n$ has radius of convergence $R > 0$, then $\forall r < R$, $\sum_{n=0}^{\infty} C_n (z - z_0)^n$ converges uniformly in $\overline{D(z_0; r)}$.

④ If a seq of continuous functions $\{f_n : [a,b] \to \mathbb{R}\}_{n=1}^{\infty}$ converges uniformly in $[a,b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$

Prop 4.11 (line-integral ver of ④)

Suppose $\{f_n\}$ is a seq. of continuous functions and $f_n \to f$ uniformly on a piecewise C' curve C. Then

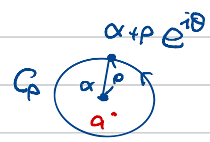
$$\lim_{n\to\infty} \int_C f_n(z)\,dz = \int_C f(z)\,dz$$

pf

$\forall \varepsilon > 0$ $\exists N$ s.t. $|f_n(z) - f(z)| < \varepsilon$ $\forall n > N$, $\forall z \in C$

$\Rightarrow$ $\left| \int_C f_n(z)dz - \int_C f(z)dz \right| = \left| \int_C f_n(z) - f(z)\,dz \right| \underset{\text{M-L}}{\leq} \varepsilon \cdot \text{length}(C)$ $\quad \forall n > N$ #

Cauchy integral formula



Lemma 5.4

Let $\alpha, a \in \mathbb{C}$, $\rho > 0$, and C_ρ be the circle $\alpha + \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

Suppose $|a - \alpha| < \rho$. Then

$$\int_{C_\rho} \frac{1}{z-a}\,dz = 2\pi i$$

pf

1° Note that

$$\int_{C_\rho} \frac{1}{z-\alpha}\,dz = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\,d\theta = 2\pi i \quad \text{and} \quad \int_{C_\rho} \frac{1}{(z-\alpha)^{k+1}}\,dz = 0,\ k = 1, 2, 3, \cdots.$$

2°. Write

$$\frac{1}{z-a} = \frac{1}{(z-\alpha) - (a-\alpha)} = \frac{1}{z-\alpha} \cdot \frac{1}{1 - \frac{a-\alpha}{z-\alpha}} = \frac{1}{z-\alpha} \cdot \frac{1}{1-\omega}$$

where $\omega = \frac{a-\alpha}{z-a}$ has fixed modulus $|\omega| = \frac{|a-\alpha|}{\rho} < 1$ $\quad \forall z \in C_\rho$

3° Since $\sum_{n=0}^{\infty} \left| \frac{1}{z-\alpha} \cdot \omega^n \right| = \sum_{n=0}^{\infty} \frac{1}{\rho} \left(\boxed{\frac{|a-\alpha|}{\rho}} \right)^n$ <1, M-test implies

$$\frac{1}{z-\alpha} \cdot \frac{1}{1-\omega} = \sum_{n=0}^{\infty} \frac{1}{z-\alpha} \omega^n = \sum_{n=0}^{\infty} \frac{1}{z-\alpha} \left(\frac{a-\alpha}{z-\alpha} \right)^n$$

converges to $\frac{1}{z-a}$ uniformly on C_ρ

$$\Rightarrow \int_{C_\rho} \frac{1}{z-a}\,dz = \int_{C_\rho} \sum_{n=0}^{\infty} \frac{1}{z-\alpha} \left(\frac{a-\alpha}{z-\alpha} \right)^n dz = \sum_{n=0}^{\infty} \int_{C_\rho} \frac{(a-\alpha)^n}{(z-\alpha)^{n+1}}\,dz$$

$$= \sum_{n=0}^{\infty} (a-\alpha)^n \int_{C_\rho} \frac{1}{(z-\alpha)^{n+1}}\,dz$$

by 1°

$$= 2\pi i$$

#