pf

Let $f = u + iv$.

$$\int_C f(z)dz = \int_a^b (u(z(t)) + i v(z(t)))\cdot(x'(t) + i y'(t))\,dt$$

change of variables for integration $t = \lambda(s)$, $dt = \lambda'(s)ds$

$$= \int_a^b u(z(t))x'(t) - v(z(t))\cdot y'(t)\,dt + i\int_a^b v(z(t))\,x'(t) + u(z(t))\,y'(t)\,dt$$

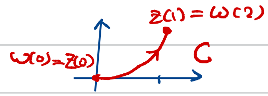
$$= \int_c^d [u(z(\lambda(s)))\,x'(\lambda(s)) - v(z(\lambda(s)))\,y'(\lambda(s))]\,\lambda'(s)\,ds + i\int_c^d [v(z(\lambda(s)))\,x'(\lambda(s)) + u(z(\lambda(s)))\,y'(\lambda(s))]\,\lambda'(s)\,ds$$

$$= \int_c^d f(z(\lambda(s)))\cdot z'(\lambda(s))\cdot\lambda'(s)\,ds \qquad \#$$

Remark (See Prop 4.5)

The above proposition ⇒ If $z(t)$, $\omega(s)$ are 2 parametrization of the same curve C with the same orientation, then $\int_a^b f(z(t))\,z'(t)dt = \int_C f(z)dz = \int_c^d f(\omega(s))\,ds$ $(\omega(s) = z(\lambda(s)))$

e.g. $z(t) = t + it^2$, $t\in[0,1]$, $\omega(s) = \frac{1}{2}s + i\cdot\frac{s^2}{4}$, $s\in[0,2]$, $f(z) = z$



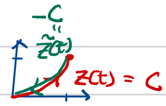
$$\int_C f(z)dz = \int_0^1 (t+it^2)(1+i\,2t)\,dt = \int_0^1 t - 2t^3 + i(t^2+2t^2)\,dt = \frac{1}{2} - \frac{2}{4} + i = i$$

$$= \int_0^2 \left(\frac{s}{2} + i\frac{s^2}{4}\right)\left(\frac{1}{2} + i\frac{s}{2}\right)ds = \int_0^2 \left(\frac{s}{4} - \frac{s^3}{8}\right) + i\left(\frac{s^2}{8} + \frac{s^2}{4}\right)ds = \frac{4}{4\cdot 2} - \frac{2^4}{8\cdot 4} + i\left(\frac{2^3}{8}\right) = i$$

Def 4.6

Suppose C is given by $z(t)$, $a\le t\le b$. Then $-C$ is defined by $z(b+a-t)$, $a\le t\le b$

e.g. $z(t) = t + it^2$, $t\in[0,1]$, $\tilde{z}(t) = z(1+0-t) = 1-t+i(1-t)^2$, $t\in[0,1]$



Prop 4.7

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

pf

$$\int_{-C} f(z)dz = \int_a^b f(z(b+a-t))\,\frac{d}{dt}(z(b+a-t))\,dt$$

$$= \int_a^b f(z(b+a-t))\,z'(b+a-t)\,(-1)\,dt$$

$s = b+a-t$, $ds = -dt$

$$= \int_b^a f(z(s))\,z'(s)\cdot ds = -\int_a^b f(z(s))\,z'(s)ds = -\int_C f(z)dz \qquad \#$$

e.g. $z(t) = t+it^2$, $t\in[0,1]$, $\tilde{z}(t) = z(1+0-t) = 1-t+i(1-t)^2$, $t\in[0,1]$, $f(z) = z$

$$\int_{-C} z\,dz = \int_0^1 [1-t+i(1-t)^2][-1+i(-2+2t)]\,dt = \int_0^1 t-1+2(1-t)^3 + i(-3)(1-t)^2\,dt = \int_1^0 [-s+2s^3+i(-3s^2)]\,(-ds)$$

$$= -\frac{1}{2} + \frac{2}{4} + i(-1) = -i$$

$$-\int_C z\,dz = -\int_0^1 (t+it^2)(1+i\cdot 2t)\,dt = -i$$

Prop 4.8

Let C be a piecewise C^1 curve, f and g be continuous on C and $\alpha,\beta\in\mathbb{C}$. Then

$$\int_C \alpha f(z) + \beta g(z)\,dz = \alpha\int_C f(z)dz + \beta\int_C g(z)dz$$

pf: exer

Example (p.48)

① $f(z) = x^2 + iy^2$, $C: z(t) = t+it$, $0\le t\le 1$ ⇒ $z'(t) = 1+i$

$$\int_C f(z)dz = \int_C x^2 dz + i\int_C y^2 dz = \int_0^1 t^2(1+i)dt + i\int_0^1 t^2(1+i)dt$$

$$= (1+i)(1+i)\int_0^1 t^2 dt = (1-1+2i)\frac{1}{3} = \frac{2i}{3}$$

② $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$, $C: z(t) = R\cos t + iR\sin t$, $0\le t\le 2\pi$, $R\ne 0$

$$\int_C f(z)dz = \int_0^{2\pi} \left(\frac{\cancel{R}\cos t}{\cancel{R^2}} - i\frac{\cancel{R}\sin t}{\cancel{R^2}}\right)(-\cancel{R}\sin t + i\cancel{R}\cos t)\,dt$$

$$= \int_0^{2\pi} 2\cos t\sin t + i(\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi}\sin 2t\,dt + 2\pi i = \underbrace{-\frac{\cos 2t}{2}\Big|_0^{2\pi}}_{=0} + 2\pi i = 2\pi i$$

③ $f(z) \equiv 1$, C: any piecewise C^1 curve

$$\int_C f(z)dz = \int_a^b z'(t)dt = z(b) - z(a)$$

M-L inequality

Lemma 4.9

Suppose $G: [a,b]\to\mathbb{C}$ is a continuous complex-valued function. Then

$$\left|\int_a^b G(t)\,dt\right| \le \int_a^b |G(t)|\,dt$$

pf

Suppose

$$\int_a^b G(t)\,dt = Re^{i\theta}, \quad R \geq 0.$$

Prop4.8 $\Rightarrow$

$$\int_a^b e^{-i\theta} G(t)\,dt = R \quad \text{——} \quad \circledast$$

Suppose $e^{-i\theta}G(t) = A(t) + iB(t)$, $A, B : [a,b] \to \mathbb{R}$. Then by $\circledast$

$$\int_a^b A(t)\,dt = R, \qquad \int_a^b B(t)\,dt = 0$$

$\Rightarrow$

$\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z|$

$$R = \left|\int_a^b G(t)\,dt\right| = \int_a^b \mathrm{Re}(e^{-i\theta}G(t))\,dt \leq \int_a^b |e^{-i\theta}G(t)|\,dt = \int_a^b |G(t)|\,dt \qquad \#$$

Recall

Let C be a piecewise C^1 curve given by $z(t)$, $t \in [a,b]$. Then the length of C is

$$L = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Thm4.10 (M-L formula)

Suppose (i) C is a piecewise C^1 curve of length L, (ii) f is continuous on C and (iii) $|f(z)| \leq M$ on C.

Then

$$\left|\int_C f(z)\,dz\right| \leq ML$$

pf

Suppose C is given by $z(t)$, $a \leq t \leq b$. Then

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\cdot z'(t)\,dt\right| \leq \int_a^b \underbrace{|f(z(t))|}_{\leq M}\cdot|z'(t)|\,dt$$

$$\leq \int_a^b M\cdot|z'(t)|\,dt = ML \qquad \#$$

(exer: Prop4.11, move Prop4.11 to next chapter)

Line integral verison of fundamental thm of Calculus

Prop4.12

Suppose $F(z)$ is an analytic on a piecewise C^1 curve C which is given by $z(t)$, $a \leq t \leq b$. Then

$$\int_C F'(z)\,dz = F(z(b)) - F(z(a))$$

pf

Let $F(z) = U(z) + iV(z)$, $\quad \gamma(t) = F(z(t)) = U(x(t)+iy(t)) + iV(x(t)+iy(t))$, $\quad a \leq t \leq b$.

$$\Rightarrow \gamma'(t) = \frac{d}{dt}\big(U(x(t)+iy(t))\big) + i\frac{d}{dt}\big(V(x(t)+iy(t))\big)$$

chain rule for real fn $\rightarrow$

$$= U_x(z(t))\cdot x'(t) + U_y(z(t))\cdot y'(t) + i\big(V_x(z(t))\cdot x'(t) + V_y(z(t))\cdot y'(t)\big)$$

C-R $\rightarrow$

$$= U_x(z(t))\cdot x'(t) - V_x(z(t))\cdot y'(t) + iV_x(z(t))\cdot x'(t) + iU_x(z(t))\cdot y'(t)$$

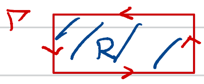
$$= \big(U_x(z(t)) + iV_x(z(t))\big)\cdot\big(x'(t) + iy'(t)\big)$$

$$= F'(z(t))\cdot z'(t)$$

$$\Rightarrow \int_C F'(z)\,dz = \int_a^b F'(z(t))\cdot z'(t)\,dt = \int_a^b \gamma'(t)\,dt = \gamma(b) - \gamma(a)$$

$$= F(z(b)) - F(z(a)) \qquad \#$$

Rectangle theorems



In this section, we assume Γ is the boundary of a rectangle R

$R = \{x+iy : a \leq x \leq b,\ c \leq y \leq d\}$

Lemma (p52)

If f is a linear function, then

$$\int_\Gamma f(z)\,dz = 0$$

pf

Let $f(z) = \alpha + \beta z$, $\quad \Gamma$ be given by $z(t)$, $a \leq t \leq b$. and $F(z) = \alpha z + \frac{\beta}{2}z^2$

$\Rightarrow F'(z) = f(z)$

Prop4.12 $\Rightarrow$

$$\int_\Gamma f(z)\,dz = \int_\Gamma F'(z)\,dz = F(z(b)) - F(z(a)) = 0 \qquad \#$$

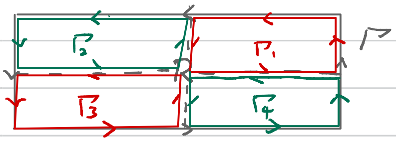
Thm 4.14 (Rectangle Thm, also see Thm 6.1)

Suppose f is analytic on $U \underset{\text{open}}{\subseteq} \mathbb{C}$ and $R \subseteq U$. Then

$$\int_\Gamma f(z)\,dz = 0$$

pf

By Prop 4.7, we may assume Γ is counterclockwise



Let $I = \int_\Gamma f(z)\,dz$.

1° We split R into 4 congruent subrectangles

By Prop 4.7, we have

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\Gamma_j} f(z)\,dz$$

Let $\Gamma^{(1)}$ be the one of the boundaries s.t.

$$\left|\int_{\Gamma^{(1)}} f(z)\,dz\right| = \max_{j=1\cdots 4} \left\{ \left|\int_{\Gamma_j} f(z)\,dz\right| \right\}$$

Since

$$|I| = \left|\int_\Gamma f(z)\,dz\right| \le \sum_{j=1}^{4} \left|\int_{\Gamma_j} f(z)\,dz\right| \le 4\left|\int_{\Gamma^{(1)}} f(z)\,dz\right|$$

we have

$$\left|\int_{\Gamma^{(1)}} f(z)\,dz\right| \ge \frac{|I|}{4}$$

Let $R^{(1)}$ be the rectangle bounded by $\Gamma^{(1)}$.

2° Repeating this process to $R^{(1)}$, we get a seq of rectangles

$$R \supset R^{(1)} \supset R^{(2)} \supset \cdots$$

and their boundaries

$$\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \cdots$$

s.t. ① $\operatorname{diam} R^{(k+1)} = \frac{1}{2} \operatorname{diam} R^{(k)}$

→ ② $\left|\int_{\Gamma^{(k)}} f(z)\,dz\right| \ge \frac{|I|}{4^k}$

3° Since each $R^{(k)}$ is compact and nonempty, by the nested property of compact sets, there exists

$$z_0 \in \bigcap_{k=1}^{\infty} R^{(k)}$$

Let $\varepsilon_z = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$. Then

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \varepsilon_z \cdot (z-z_0)$$

and, by the analyticity of f at z_0,

$$\varepsilon_z \longrightarrow 0 \quad \text{as } z \to z_0$$

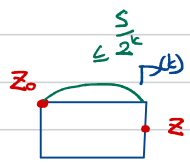
4° By Lemma,

linear

→ $$\int_{\Gamma^{(k)}} f(z)\,dz = \int_{\Gamma^{(k)}} \underline{f(z_0) + f'(z_0)(z-z_0)} + \varepsilon_z \cdot (z-z_0)\,dz$$

$$= \int_{\Gamma^{(k)}} \varepsilon_z \cdot (z-z_0)\,dz$$

Let the length of the largest side of Γ $= s$.



Note that for $z \in \Gamma^{(k)}$,

$$|z - z_0| \le \frac{\sqrt{2}}{2^k} s$$

So $\forall \varepsilon > 0$, $\exists N$ s.t.

$$|\varepsilon_z - 0| < \varepsilon \qquad \forall\ |z - z_0| \le \frac{\sqrt{2}}{2^N} s$$

$\Rightarrow \forall k \ge N$, by Prop 4.10, $\left(z \in \Gamma^{(k)} \Rightarrow |z-z_0| \le \frac{\sqrt{2}}{2^k} s \le \frac{\sqrt{2}}{2^N} s \Rightarrow |\varepsilon_z| < \varepsilon\right)$

$$\frac{|I|}{4^k} \le \left|\int_{\Gamma^{(k)}} f(z)\,dz\right| = \left|\int_{\Gamma^{(k)}} \varepsilon_z \cdot (z-z_0)\,dz\right| \le \varepsilon \cdot \frac{\sqrt{2}}{2^k} s \cdot \operatorname{length}(\Gamma^{(k)})$$

$$\le \varepsilon \cdot \frac{\sqrt{2}}{2^k} s \cdot \frac{s}{2^k} \cdot 4 = \varepsilon \cdot \frac{4\sqrt{2}}{4^k} s^2$$

indep of N, k. $\quad$ ε is arbitrary

$\Rightarrow \forall \varepsilon > 0$, $\exists N$ s.t. $\forall k \ge N$,

$$\frac{|I|}{4^k} \le \varepsilon \cdot 4\sqrt{2}\, s^2 \frac{1}{4^k} \iff |I| \le \varepsilon \cdot 4\sqrt{2}\, s^2$$

$\Rightarrow I = 0$ #

Thm5.1 (Rectangle Thm II, also see Thm 6.1)

Suppose f is analytic on $U \underset{\text{open}}{\subseteq} \mathbb{C}$, and $R \subseteq U$. For $a \in U$, let

$$g(z) := \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \in U \setminus \{a\} \\ f'(a), & z = a. \end{cases}$$

Then $\int_\Gamma g(z)\,dz = 0$

pf

case1 $a \notin R$

Note that $\frac{1}{z-a}$ is analytic on $\mathbb{C} \setminus \{a\} \supseteq R$ $\Rightarrow$ $g = \frac{f(z)-f(a)}{z-a}$ is also analytic on $U \setminus \{a\} \supseteq R$

$\Rightarrow$ by Thm4.14, $\int_\Gamma g(z)\,dz = 0$

case2 $a \in \Gamma = \partial R$

compact

Note that g is continuous on R $\Rightarrow$ $\exists M>0$ s.t. $|g(z)| \le M$ $\forall z \in R$

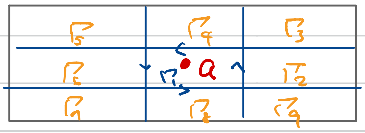
We divide R into 6 subrectangles:

$$\Rightarrow \int_\Gamma g(z)\,dz = \sum_{j=1}^{6} \int_{\Gamma_j} g(z)\,dz \underset{\text{by case1}}{=} \int_{\Gamma_1} g(z)\,dz$$

Note that $\forall \varepsilon>0$, $\exists$ division s.t. $\text{length}(\Gamma_1) < \varepsilon$

$\Rightarrow$ $\left|\int_\Gamma g(z)\,dz\right| = \left|\int_{\Gamma_1} g(z)\,dz\right| \le M \cdot \varepsilon$ $\quad \forall \varepsilon > 0$ $\quad \Rightarrow \int_\Gamma g(z)\,dz = 0$

case3 $a \in \text{int}(R)$



Similar as case2, we divide R:

$\forall \varepsilon>0$, $\exists$ such division$_9$ s.t. $\text{length}(\Gamma_1) < \varepsilon$

$\Rightarrow$ $\left|\int_\Gamma g(z)\,dz\right| = \left|\sum_{j=1}^{9} \int_{\Gamma_j} g(z)\,dz\right| \overset{\text{case1}}{=} \left|\int_{\Gamma_1} g(z)\,dz\right| \le M \cdot \varepsilon$ $\quad \forall \varepsilon>0$ $\quad \Rightarrow \int_\Gamma g(z)\,dz = 0$ #

Closed curve thm

Def4.13

A curve C given by $z(t)$, $a \le t \le b$, is called closed if $z(a) = z(b)$.

A closed curve C is called a simple closed curve if $\nexists$ other points coincide, i.e., $z(s)=z(t) \Rightarrow \{s,t\}=\{a,b\}$

e.g. ①  is closed, but not simple closed ②  is simple closed

Thm4.15 (Integral Thm, also see Cor6.2, Thm6.2)

Suppose $r \in (0,\infty]$. If $f: D(z_0;r) \overset{\subseteq \mathbb{C}}{\longrightarrow} \mathbb{C}$ is analytic, then $\exists$ analytic functions F, G on $D(z_0;r)$

s.t. $F'(z) = f(z)$ and $G'(z) = g(z)$ $\quad \forall z \in D(z_0;r)$,

where $$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \in D(z_0;r) \setminus \{a\} \\ f'(a), & z = a \end{cases}$$

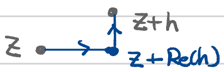
Notation: $D(z_0;\infty) = \mathbb{C}$

pf

We will construct $F(z)$ by using Thm4.14. The construction of $G(z)$ is similar by applying Thm5.1.

Notation:

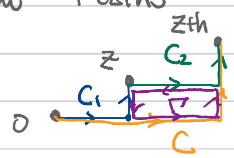
$$\int_z^{z+h} f(\zeta)\,d\zeta = \int_C f(\zeta)\,d\zeta, \qquad C =$$



Special case: $z=0$

Let $F(z) := \int_0^z f(\zeta)\,d\zeta$

Note that $F(z+h) = F(z) + \int_z^{z+h} f(\zeta)\,d\zeta$ because



$$F(z+h) - \left(F(z) + \int_z^{z+h} f(\zeta)\,d\zeta\right) = \int_C f(\zeta)\,d\zeta - \left(\int_{C_1} f(\zeta)\,d\zeta + \int_{C_2} f(\zeta)\,d\zeta\right)$$

$$= \int_\Gamma f(\zeta)\,d\zeta = 0 \quad \text{(by Thm4.14)}$$