Ch4 Line integrals

Introduction

Recall: by Thm 2.9, if $\sum c_n z^n$ converges $\forall z \in \mathbb{C}$, then $f(z) = \sum c_n z^n$ is an entire function

A big goal of Ch4-5: if $f(z)$ is entire, then $f(z) = \sum c_n z^n$ $\forall z \in \mathbb{C}$ ($\Leftrightarrow$ f is ∞ differentiable)

Main Tool: line integrals and ** Cauchy integral formula (Thm 5.3)

Line integrals

Def 4.1

Let $f: [a,b] \xrightarrow{\text{c/R}} \mathbb{C}$ be a continuous function. Suppose

$$f(t) = u(t) + i\,v(t) \qquad t \in [a,b]$$

where $u, v: [a,b] \to \mathbb{R}$. Define

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

eg. $\int_0^1 t + it^2\,dt = \int_0^1 t\,dt + i\int_0^1 t^2\,dt = \frac{1}{2} + \frac{1}{3}i$

Recall

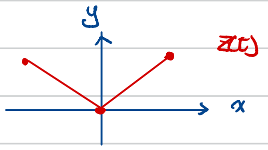
A function $\alpha: [a,b] \to \mathbb{R}$ is C^1 if α is continuous on $[a,b]$, differentiable on (a,b), and α' is continuous on (a,b)

Def 4.2 (the authors use "piecewise differentiable", "smooth" in book)

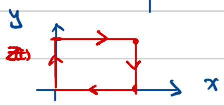
Let $z(t) = x(t) + iy(t)$, $a \le t \le b$, be a curve on $\mathbb{C}$. The curve is called piecewise C^1 if $x(t)$ and $y(t)$ are continuous on $[a,b]$ and $\exists\ a < t_1 < t_2 < \cdots < t_n < b$ s.t. the restrictions of x, y to intervals $[a, t_1], [t_1, t_2], \cdots, [t_{n-1}, b]$ are C^1.

Example

① $C_1 = z(t) := \begin{cases} t + it, & t \in [0, +1] \\ t - it, & t \in [-1, 0] \end{cases}$ is a piecewise C^1 curve



② $C_2 = z(t) := \begin{cases} it & t \in [0,1] \\ (t-1) + i & t \in [1,2] \\ 1 + (3-t)i & t \in [2,3] \\ 4 - t & t \in [3,4] \end{cases}$ is a piecewise C^1 curve



Recall (change of variables for integration)

Suppose $\lambda: [a,b] \to \mathbb{R}$ is C^1, $f: \mathbb{R} \to \mathbb{R}$ is continuous. Then

$$\int_a^b f(\lambda(x))\,\lambda'(x)\,dx = \int_{\lambda(a)}^{\lambda(b)} f(u)\,du$$

Remark (line integral in Calculus)

$\int_C F(r) \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$

("work" in physics) inner product!! different!

Def 4.3

Let C be a piecewise C^1 curve given by $z(t) = x(t) + iy(t)$, $a \le t \le b$, and suppose $f: U \xrightarrow{\mathbb{C}\text{ open}} \mathbb{C}$ is defined and continuous at all points in C. Then the (complex) integral of f along C is

$$\int_C f(z)\,dz := \int_a^b f(z(t)) \cdot z'(t)\,dt = \int_a^b f(z(t))\,(x'(t) + iy'(t))\,dt$$

complex product!!

Example

① $f(z) = z^2$, $C_1 = C_1$ in previous example

$$\int_{C_1} f(z)\,dz = \int_{-1}^1 (z(t))^2 \cdot z'(t)\,dt = \int_{-1}^0 \overbrace{(t-it)^2}^{t^2 - t^2 - 2it^2 = -2it^2} \cdot (1-i)\,dt + \int_0^1 \overbrace{(t+it)^2}^{t^2 - t^2 + 2it^2 = 2it^2}(1+i)\,dt$$

$$= \int_{-1}^0 -2it^2 - 2t^2\,dt + \int_0^1 2it^2 - 2t^2\,dt = (-2i-2)\frac{1}{3} + (2i-2)\frac{1}{3} = -\frac{4}{3}$$

② $f(z) = z^2$, $C_2 = C_2$ in previous example

$$\int_{C_2} f(z)\,dz = \int_0^1 (it)^2\,i\,dt + \int_1^2 (t-1+i)^2\,dt + \int_2^3 (1 + (3-t)i)^2 (-i)\,dt + \int_3^4 (4-t)^2(-1)\,dt$$

$\int_1^2 (t-1+i)^2\,dt = \int_0^1 (t+i)^2\,dt = \int_0^1 t^2 + 2it - 1\,dt$

$\int_2^3 (1 - (3-t)^2 + 2(3-t)i)(-i)\,dt = \int_1^0 (1 - s^2 + 2s\,i)\,i\,ds = \int_0^1 2s + i(s^2 - 1)\,ds$

$\int_3^4 (4-t)^2(-1)\,dt = \int_1^0 s^2\,ds = -\frac{1}{3}$

$$= (-i \cancel{-\tfrac{1}{3}}) + (\cancel{\tfrac{4}{3}} \cancel{+} \cancel{+i}) + (\cancel{+} \cancel{\tfrac{1}{3}i} \cancel{-i}) + (\cancel{-\tfrac{4}{3}}) = 0$$

Remark for ②: we will show that if $z(a) = z(b)$ and f is entire, then $\int_C f(z)\,dz = 0$ (Thm 4.16 = closed curve thm)

Prop (Change of variables)

Suppose $\lambda: [c,d] \to [a,b]$ is C^1 s.t. $\lambda(c) = a$, $\lambda(d) = b$. Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_c^d f(z(\lambda(s)))\,\underbrace{z'(\lambda(s))\,\lambda'(s)}_{= \frac{d}{ds}(z(\lambda(s)))}\,ds$$

pf

Let $f = u + iv$.

$$\int_C f(z)dz = \int_a^b (u(z(t)) + i v(z(t)))\cdot(x'(t) + i y'(t))\,dt$$

change of variables for integration $t=\lambda(s)$, $dt = \lambda'(s)ds$

$$= \int_a^b u(z(t))x'(t) - v(z(t))\cdot y'(t)\,dt + i\int_a^b v(z(t))x'(t) + u(z(t))y'(t)\,dt$$

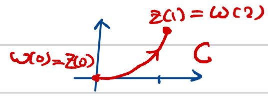
$$= \int_c^d [u(z(\lambda(s)))x'(\lambda(s)) - v(z(\lambda(s)))y'(\lambda(s))]\lambda'(s)ds + i\int_c^d [v(z(\lambda(s)))x'(\lambda(s)) + u(z(\lambda(s)))y'(\lambda(s))]\lambda'(s)ds$$

$$= \int_c^d f(z(\lambda(s)))\cdot z'(\lambda(s))\cdot \lambda'(s)\,ds \qquad \#$$

Remark (See Prop 4.5)

The above proposition $\Rightarrow$ If $z(t)$, $w(s)$ are 2 parametrization of the same curve C with the same orientation, then $\int_a^b f(z(t))z'(t)dt = \int_C f(z)dz = \int_c^d f(w(s))w'(s)ds$ ($w(s) = z(\lambda(s))$)

e.g. $z(t) = t + it^2$, $t\in[0,1]$, $w(s) = \frac{1}{2}s + i\cdot\frac{s^2}{4}$, $s\in[0,2]$, $f(z)=z$



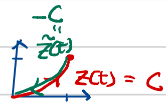
$$\int_C f(z)dz = \int_0^1 (t+it^2)(1+i2t)dt = \int_0^1 t - 2t^3 + i(t^2+2t^2)dt = \frac{1}{2} - \frac{2}{4} + i = i$$

$$= \int_0^2 \left(\frac{s}{2} + i\frac{s^2}{4}\right)\left(\frac{1}{2} + i\frac{s}{2}\right)ds = \int_0^2 \left(\frac{s}{4} - \frac{s^3}{8}\right) + i\left(\frac{s^2}{8} + \frac{s^2}{4}\right)ds = \frac{4}{4\cdot 2} - \frac{2^4}{8\cdot 4} + i\left(\frac{2^3}{8}\right) = i$$

Def 4.6

Suppose C is given by $z(t)$, $a\le t\le b$. Then $-C$ is defined by $z(b+a-t)$, $a\le t\le b$

e.g. $z(t) = t + it^2$, $t\in[0,1]$, $\tilde{z}(t) = z(1+0-t) = 1-t+i(1-t)^2$, $t\in[0,1]$



Prop 4.7

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

pf

$$\int_{-C} f(z)dz = \int_a^b f(z(b+a-t))\frac{d}{dt}(z(b+a-t))\,dt$$

$$= \int_a^b f(z(b+a-t))\,z'(b+a-t)\,(-1)dt$$

$s = b+a-t$, $ds = -dt$

$$= \int_b^a f(z(s))\,z'(s)\cdot ds = -\int_a^b f(z(s))z'(s)ds = -\int_C f(z)dz \qquad \#$$

e.g. $z(t) = t+it^2$, $t\in[0,1]$, $\tilde{z}(t) = z(1+0-t) = 1-t+i(1-t)^2$, $t\in[0,1]$, $f(z) = z$

$$\int_{-C} z\,dz = \int_0^1 [1-t+i(1-t)^2][-1+i(-2+2t)]\,dt = \int_0^1 t-1+2(1-t)^3 + i(-3)(1-t)^2\,dt = \int_1^0 [-s+2s^3+i(-3s^2)](-ds)$$

$$= -\frac{1}{2} + \frac{2}{4} + i(-1) = -i$$

$$-\int_C z\,dz = -\int_0^1 (t+it^2)(1+i\cdot 2t)\,dt = -i$$

Prop 4.8

Let C be a piecewise C' curve, f and g be continuous on C and $\alpha,\beta\in\mathbb{C}$. Then

$$\int_C \alpha f(z) + \beta g(z)\,dz = \alpha\int_C f(z)dz + \beta\int_C g(z)dz$$

pf: exer

Example (p.48)

① $f(z) = x^2 + iy^2$, C: $z(t) = t+it$, $0\le t\le 1$ $\Rightarrow$ $z'(t) = 1+i$

$$\int_C f(z)dz = \int_C x^2dz + i\int_C y^2 dz = \int_0^1 t^2(1+i)dt + i\int_0^1 t^2(1+i)dt$$

$$= (1+i)(1+i)\int_0^1 t^2dt = (1-1+2i)\frac{1}{3} = \frac{2i}{3}$$

② $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ C: $z(t) = R\cos t + iR\sin t$, $0\le t\le 2\pi$, $R\ne 0$

$$\int_C f(z)dz = \int_0^{2\pi}\left(\frac{\cancel{R}\cos t}{R^{\cancel{2}}} - i\frac{\cancel{R}\sin t}{R^{\cancel{2}}}\right)(-\cancel{R}\sin t + i\cancel{R}\cos t)\,dt$$

$$= \int_0^{2\pi} 2\cos t\sin t + i(\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi}\sin 2t\,dt + 2\pi i = \underbrace{\cancel{-\frac{\cos 2t}{2}\Big|_0^{2\pi}}}_{=0} + 2\pi i = 2\pi i$$

③ $f(z)\equiv 1$, C: any piecewise C' curve

$$\int_C f(z)dz = \int_a^b z'(t)dt = z(b) - z(a)$$

M-L inequality

Lemma 4.9

Suppose $G: [a,b]\to\mathbb{C}$ is a continuous complex-valued function. Then

$$\left|\int_a^b G(t)\,dt\right| \le \int_a^b |G(t)|\,dt$$