Analyticity

Def 3.3

f is analytic at z if f is differentiable in a nbd of z

f is analytic on a set S if f is differentiable in a nbd of S

f is called an entire function if f is differentiable in whole $\mathbb{C}$

HW: Prop 3.5

Note: analyticity implies a lot of consequences. The following are 2 examples.

Prop 3.6

If $f = u + iv$ is analytic in a region (i.e. open cnt subset in $\mathbb{C}$) and u is constant, then f is constant.

pf

$u = \text{Const} \Rightarrow u_x = u_y = 0 \overset{\text{C-R}}{\Rightarrow} v_x = v_y = 0 \Rightarrow v$ is also const $\Rightarrow f$ is const. #

Prop 3.7

If f is analytic in a region D and $|f|$ is constant in D, then f is constant.

pf

If $|f| = 0$, then $f \equiv 0$.

Assume $u^2 + v^2 = C \neq 0$

take ∂_x, ∂_y — C-R eq

$$\Rightarrow \begin{cases} u u_x + v v_x = 0 = u u_x - v u_y & ① \\ u u_y + v v_y = 0 = u u_y + v u_x & ② \end{cases}$$

$\Rightarrow u\cdot① + v\cdot②: \ (u^2+v^2) u_x = 0 = C \cdot u_x \Rightarrow u_x \equiv 0 = v_y$

$u\cdot② - v\cdot①: \ (u^2+v^2) u_y = 0 = C \cdot u_y \Rightarrow u_y \equiv 0 = -v_x$

$\Rightarrow f$ is constant in D. #

Examples of entire function

① $f(z) = z \Rightarrow f'(z) = \lim_{h\to 0} \frac{(z+h) - z}{h} = 1$ ← skip

② By ①, ② Prop 2.5, a polynomial $P(z) = a_0 + a_1 z + \cdots + a_N z^N$ is differentiable at all $z \in \mathbb{C}$ and

$P'(z) = a_1 + 2a_2 z + \cdots + N a_N z^{N-1}$ ← Prop 2.6

③ Exponential map: for $z = x + iy \in \mathbb{C}$, we define

$$e^z = e^{x+iy} := e^x(\cos y + i \sin y)$$

Lemma

$f(z) = e^z$ is entire

pf

Write $f(z) = u(x,y) + i v(x,y)$ where $u = e^x \cos y$, $v = e^x \sin y$

Since $f_x = u_x + i v_x = e^x \cos y + i e^x \sin y$ are continuous at any $z = x+iy \in \mathbb{C}$

$f_y = u_y + i v_y = -e^x \sin y + i e^x \cos y$

and

$f_y = i(e^x \cos y + i e^x \sin y) = i f_x$ ← C-R eq

we conclude that $f(z) = e^z$ is differentiable at all $z \in \mathbb{C}$. #

Prop

- $e^{z_1+z_2} = e^{z_1} e^{z_2}$
- $e^{-z} = \frac{1}{e^z}$
- $|e^z| = e^x$
- $e^0 = 1$
- if $x \in \mathbb{R}$, e^x = the exponential defined in Calculus
- $(e^z)' = (e^z)_x = e^z$

exer Prove $f(z) = e^z$ is the unique entire function s.t. $f(x) = e^x$ $\forall x \in \mathbb{R}$ and $f(z_1+z_2) = f(z_1) f(z_2)$ $\forall z_1, z_2 \in \mathbb{C}$

④ Trigonometric function: for $z \in \mathbb{C}$, we define

$\sin z := \frac{1}{2i}(e^{iz} - e^{-iz})$ ← entire because they're compositions and sums of entire functions

$\cos z := \frac{1}{2}(e^{iz} + e^{-iz})$

exer $(\sin z)' = \cos z$, $\sin^2 z + \cos^2 z = 1$, ...

Power series (good for constructing analytic functions)

A power series in z (center at $0 \in \mathbb{C}$) is a series of the form
$$\sum_{k=0}^{\infty} c_k z^k, \quad c_k \in \mathbb{C}$$

We say $\sum c_k z^k$ converges (resp diverges) (for z_0) if the seq $\{\sum_{k=0}^{n} c_k z_0^k\}_{n=0}^{\infty}$ Converges (resp diverges). The limit is denoted by $\sum_{k=0}^{\infty} c_k z_0^k$.

Recall that for real-valued seq $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$, its limsup is
$$\overline{\lim_{n\to\infty}} a_n = \limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_k : k \ge n\}$$ ← this part is non-increasing $\Rightarrow$ $\overline{\lim}$ = a real number or $\pm\infty$

The following theorems can be proved by adapting arguments in Calculus.

Thm 2.8 (root test)

Suppose $L = \overline{\lim_{n\to\infty}} |c_n|^{1/n}$

① If $L = 0$, $\sum c_n z^n$ converges for all $z \in \mathbb{C}$.

② If $L = \infty$, $\sum c_n z^n$ converges for $z = 0$ only

③ If $0 < L < \infty$, set $R = \frac{1}{L}$. Then $\sum c_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$

Def

The number
$$R = \begin{cases} \infty & \text{if } L = 0 \\ \frac{1}{L} & \text{if } 0 < L < \infty \\ 0 & \text{if } L = \infty \end{cases}$$
is call the radius of convergence of the power series $\sum c_n z^n$

Prop (Exer 13, Ch2. This $\Rightarrow$ ratio test)

Suppose $\{a_n\}$ is a seq of positive real numbers and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

Then $\lim_{n\to\infty} a_n^{1/n} = L$.

Example

Let $\sum c_n z^n = \sum \frac{1}{n!} z^n$. Since $\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \to 0 \Rightarrow \lim_{k\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0 = \overline{\lim_{k\to\infty}} \left(\frac{1}{n!}\right)^{1/n} \Rightarrow$ $R = \infty$, $\sum \frac{1}{n!} z^n$ converges $\forall z$

Thm 2.9

Let R be the radius of convergence of $\sum c_n z^n$. Let $f(z) := \sum_{n=0}^{\infty} c_n z^n$ for $|z| < R$.

Then $f'(z)$ exists, the radius of convergence of $\sum n c_n z^{n-1} = R$, and
$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} \quad \text{for } |z| < R$$

Cor 2.10

$f(z) = \sum_{n=0}^{\infty} c_n z^n$ is infinitely differentiable in $D(0;R)$, where R = radius of convergence.

$\{z \in \mathbb{C} : |z| < R\}$ = domain of convergence

Cor 2.11

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ has a nonzero radius of convergence, then
$$c_n = \frac{f^{(n)}(0)}{n!} \quad \forall n \in \mathbb{N}$$

Example

Let $f(z) = \sum_{n=0}^{\infty} z^n$, $|z| < 1$. Note that $\left(\sum_{n=0}^{N} z^n\right)(1-z) = \sum_{n=0}^{N} z^n - \sum_{n=1}^{N+1} z^n = 1 - z^{N+1}$

So for $|z| < 1$, $f(z) = \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

$\Rightarrow f(z) = \frac{1}{1-z}$ is analytic in $D(0;1)$ and $f'(z) = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}$ for $|z| < 1$.

Thm 2.12 (Uniqueness of power series)

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Suppose $\exists$ nonzero seq $\{z_k\} \subseteq \mathbb{C}$ s.t. ① $\lim_{k\to\infty} z_k = 0$ ② $f(z_k)$ converges and $= 0$ $\forall k$

Then $c_n = 0$ $\forall n$, i.e. $f(z) \equiv 0$

pf

Since $f(z)$ converges at $z_k \neq 0$, the radius of convergence of $f(z)$ $= R > 0$.

$\underset{\text{Thm 2.9}}{\Rightarrow}$ f is differentiable at 0 $\Rightarrow$ f is continuous at 0: $f(0) = c_0 = \lim_{k\to\infty} f(z_k) = 0$

pf

Note that $\frac{f(z_k)}{z_k} = c_1 + c_2 z_k + \cdots = \sum_{n=1}^{\infty} c_n z_k^{n-1}$ also converges $\overset{|z_k|>0}{\Rightarrow}$ radius of convergence >0

$\Rightarrow g(z) := \sum_{n=1}^{\infty} c_n z^{n-1}$ is also continuous at 0

$\Rightarrow c_1 = g(0) = \lim_{k\to\infty} g(z_k) = \lim_{k\to\infty} \frac{f(z_k)}{z_k} = 0$

Similarly, if $c_j = 0 \quad \forall\, 0 \le j < n$, then

$c_n = \lim_{k\to\infty} \frac{f(z_k)}{z_k^n} = 0$

induction $\Rightarrow$ $c_n = 0 \quad \forall n$

#

Cor 2.14

recall: p is an acc pt of $S \subseteq \mathbb{C}$ if $D(p;r) \cap S \setminus \{p\} \neq \phi \quad \forall r > 0$.

If $\sum a_n z^n$ and $\sum b_n z^n$ converge and agree on a set with an accumulation point at 0, then $a_n = b_n \ \forall n$

pf

Assumption $\Rightarrow \exists$ nonzero seq $\{z_k\} \subseteq \mathbb{C}$ s.t. $\lim_{k\to\infty} z_k = 0$ and $\sum_{n=0}^{\infty} a_n z_k^n = \sum_{n=0}^{\infty} b_n z_k^n \quad \forall k$

$\Rightarrow \sum_{n=0}^{\infty} (a_n - b_n) z_k^n = 0 \quad \forall k \overset{\text{Thm 2.12}}{\Rightarrow} a_n - b_n = 0 \quad \forall n \Rightarrow a_n = b_n \ \forall n$ #

Example (proofs are outlined in homework 3)

① $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad \forall z \in \mathbb{C}$

② $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad \forall z \in \mathbb{C}$

③ $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad \forall z \in \mathbb{C}$

Remark (operations on power series, see homework 3)

Suppose $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2 respectively. Then for $|z| < \min\{R_1, R_2\}$, the power series $\sum (a_n + b_n) z^n$ and $\sum \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$ converge to the limits:

$$\sum_{n=0}^{\infty} (a_n + b_n) z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n$$

and

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) \cdot z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n z^n\right)$$

from distribution law

Example

Recall that we computed

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \quad \forall\, |z| < 1$$

by $\left(\frac{1}{1-z}\right)'$. This formula also can be computed by the method:

$$\frac{1}{(1-z)^2} = \left(\sum_{n=0}^{\infty} z^n\right) \cdot \left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} 1 \cdot 1\right) z^n = \sum_{n=0}^{\infty} (n+1) z^n = \sum_{n=1}^{\infty} n z^{n-1} \quad \forall\, |z| < 1$$

Remark

Power series in z at $\alpha \in \mathbb{C}$ are of the form

$\sum c_n (z-\alpha)^n$

One can use the previous results by the substitution $w = z - \alpha$