Ch16 Harmonic function

Def 16.1 $u: D \xrightarrow{} \mathbb{R}$, $D \overset{\text{open}}{\subset} \mathbb{R}^2$

A real-valued function $u(x,y)$ which is twice continuously differentiable and satisfies Laplace's equation

$$\Delta u := u_{xx} + u_{yy} = 0$$

throughout a domain D is said to be harmonic in D.

Thm 16.2

If $f = u + iv$ is analytic in D, then u and v are harmonic in D.

pf

By Cauchy-Riemann eq.,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{matrix} u_{xx} = v_{yx} = v_{xy} = -u_{yy} \\ v_{yy} = u_{xy} = u_{yx} = -v_{xx} \end{matrix} \quad \text{in } D$$

$\Rightarrow \Delta u = 0$, $\Delta v = 0$ in D #

Remark

The converse of Thm 16.2 is NOT true.

Example (exer 4, Ch16)

The function ← harmonic by computation

$$u(x,y) = \log(x^2+y^2)$$

is harmonic in $D = \mathbb{R}^2 \setminus \{(0,0)\}$, but is NOT the real part of an analytic function in D

pf

Note that if $z = x+iy \neq 0$, then $\operatorname{Re}\log z = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2) = \frac{1}{2}u(x,y)$.

If $f: D \to \mathbb{C}$ is analytic s.t. $\operatorname{Re} f = u$, then

$$f(z) - 2\log z$$

is analytic in $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ with the real part vanishing. $\overset{C\text{-}R}{\Rightarrow}$ $f(z) - 2\log z = iC$ ← C is a real constant

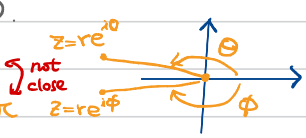
Here, $\log z$ is the analytic branch in $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ with $\log 1 = 0$.

However,

$$f(z) = 2\log z + iC$$

$\operatorname{Im} f(re^{i\theta}) = C + 2\theta \sim C + 2\pi$

$\operatorname{Im} f(re^{i\phi}) = C + 2\phi \sim C - 2\pi$



cannot be continuously extended to $\mathbb{C} \setminus \{0\}$ $(\longleftarrow\!\!\times\!\!\longleftarrow)$ #

Thm 16.3

If u is harmonic in D, then

a. u_x is the real part of an analytic function in D

b. If D is simply connected, then u is the real part of an analytic function in D

pf

a. Let $\boxed{f = u_x - iu_y}$. Since $u \in C^2(D)$, f has continuous first-order partial derivatives.

Moreover, by the harmonicity of u

$$f_y = u_{xy} - iu_{yy} = u_{yx} + iu_{xx} = if_x$$

so that f satisfies the Cauchy-Riemann eq $\overset{\text{Prop 3.2}}{\Rightarrow}$ f is analytic in D

b. If D is simply connected, by Integral Thm (Thm 8.5), $\exists$ an analytic function $F = A + iB$ in D s.t. $F' = u_x - iu_y$ ← analytic by a.

$\Rightarrow$ $F'(z) = A_x + iB_x \overset{C\text{-}R}{=} A_x - iA_y = u_x - iu_y$

$\Rightarrow$ $A_x = u_x$, $A_y = u_y$

$\Rightarrow$ $A(x,y) = u(x,y) + C$ for some constant $C \in \mathbb{R}$

$\Rightarrow$ $u = \operatorname{Re}(F(z) - C)$ #

Example

$u(x,y) = x - e^x \sin y$ is harmonic in $\mathbb{R}^2$ $= 1 + i\, e^x(\cos y + i \sin y)$

$\Rightarrow f(z) = u_x(z) - i u_y(z) = 1 - e^x \sin y + i\, e^x \cos y$ is entire

In fact, $f(z) = 1 + i e^z$ and if we set

$$F(z) = \int_0^z f(w)\, dw = z + i e^z - i,$$

then

$$u(z) = \operatorname{Re} F(z)$$ #

Mean-Value Thm for harmonic functions

Thm 16.4 (MVT)

If u is harmonic in $D(z_0; R)$, then for $0 < r < R$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta .$$

pf

Since $D(z_0;R)$ is simply connected we may assume $u = \operatorname{Re} f$ (Thm 16.3) for some f analytic in $D(z_0;R)$

By Thm 6.12 (MVT for analytic functions)

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \quad \circledast$$

By taking the real parts, $\circledast \Rightarrow$ Thm 16.4 #

Thm 16.5 (Maximum-Modulus Thm)

If u is a nonconstant harmonic function in a region D, u has no maximum or minimum points in D.

pf: By MVT or Open Mapping Thm

By C-harmonic function, we mean a continuous function which is harmonic in the interior

Thm 16.5 $\Rightarrow$ a C-harmonic function in a compact domain must assume its maximum and minimum values on the boundary of the domain.

Cor 16.6

If two C-harmonic functions u_1 and u_2 agree on the boundary of a compact domain D, then $u_1 = u_2$ throughout D.

pf

$u := u_1 - u_2$ is C-harmonic in D, so u takes its maximum and minimum on the boundary.

Since $u \equiv 0$ on ∂D, we have $0 \le u \le 0$ in D $\Rightarrow$ $u_1 \equiv u_2$ in D. #

Therefore, a C-harmonic function in a compact domain is determined by its value on the boundary of the compact domain

Next:

How can we determine the C-harmonic function explicitly by its values on the boundary?

cf: "Dirichlet Problem" ← i.e. Solve $\begin{cases} \Delta u = 0 \\ u|_{\partial D} = f \end{cases}$

Suppose we know the values of a function on the boundary of a domain. D.

Can we find a C-harmonic function whose values on ∂D are the given ones.

Thm 16.7

Suppose u is C-harmonic in $D(0;1)$. Then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, K(\theta, z)\, dz$$

uniqueness of the sol of $\begin{cases} \Delta u = 0 \\ u|_{\partial D(0;1)} = u(e^{i\theta}) \end{cases}$

where

$$K(\theta, z) := \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)$$

is called the "Poisson kernel." In polar form,

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(e^{i\theta})\,(1-r^2)}{1-2r\cos(\theta-\varphi)+r^2}\, d\theta$$

pf (sketch)

① Show that we may assume $u = \operatorname{Re} f$ for some f analytic in $\overline{D(0;1)}$

(Note: $\overline{D(0;1)}$ is closed, so it's slightly different from Thm 16.3)

② Recall that, by Cauchy Integral Formula (Thm 6.4),

$$f(z) = \frac{1}{2\pi i}\int_{|\omega|=1} \frac{f(\omega)}{\omega - z}\, d\omega = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left(\frac{e^{i\theta}}{e^{i\theta}-z}\right) d\theta \quad \text{— ⓐ}$$

③ For $z \in D(0;1)$, $\dfrac{f(\omega)}{\omega - \frac{1}{\bar{z}}}$ is analytic $\forall \omega \in D(0;1)$. By Closed Curve Thm (Thm 8.6),

$$0 = \frac{1}{2\pi i}\int_{|\omega|=1} \frac{f(\omega)}{\omega - \frac{1}{\bar{z}}}\, d\omega = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left(\frac{e^{i\theta}}{e^{i\theta} - 1/\bar{z}}\right) d\theta \quad \text{— ⓑ}$$

④ By computing $\operatorname{Re}(ⓐ - ⓑ)$, we can prove the theorem. #

Thm 16.8 (Dirichlet Problem)

Suppose $g : \{|z|=1\} = \partial D(0;1) \to \mathbb{R}$ is continuous. Then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, K(\theta, z)\, d\theta$$

existence of sol.

is the solution of

$$\begin{cases} \Delta u = 0 & \text{in } D(0;1) \\ u(e^{i\theta}) = g(e^{i\theta}) \end{cases}$$

pf: skip. See p.230–231.

Remark

By considering appropriate conformal mapping, one can solve the Dirichlet Problem for a bounded simply connected domain D.

$$\begin{cases} \Delta u = 0 \\ u|_{\partial D} = g \end{cases} \qquad D \underset{f^{-1}}{\overset{f}{\rightleftarrows}} D(0;1) = U \xrightarrow{\tilde{f}} \mathbb{C}$$

conformal mappings

Solve $\begin{cases} \Delta \tilde{u} = 0 \\ \tilde{u}|_{\partial U} = g \circ f^{-1} \end{cases}$

(Assume $f(\partial D) = \partial U$)

by Thm 16.8. Assume $\tilde{u} = \operatorname{Re}\tilde{f}$

Then $\tilde{f} \circ f$ is analytic in D

$\Rightarrow u = \operatorname{Re}(\tilde{f}\circ f) = \tilde{u}\circ f$ is harmonic in D

and $u|_{\partial D} = \operatorname{Re}(\tilde{f}\circ f|_{\partial D}) = g \circ f^{-1} \circ f = g$

i.e. $u = \tilde{u}\circ f$ is a sol.

Example

Solve $\begin{cases} \Delta u = 0 & \text{in } U = D(0;1) \\ u(x,y) = x^2 & \forall (x,y) \in \partial U \end{cases}$

sol

Note that

$$\mathrm{Re}(z^2) = x^2 - y^2$$

and if $|z|^2 = 1 = x^2+y^2$, then $\mathrm{Re}(z^2) = 2x^2 - 1$ ← want x^2 here

$\Rightarrow$ if $|z|^2 = 1$,

$$\mathrm{Re}((z^2+1)/2) = x^2$$

$\Rightarrow$ $u(x,y) = \mathrm{Re}\left(\frac{z^2+1}{2}\right) = \frac{1}{2}(x^2-y^2) + \frac{1}{2}$ is a sol. #

Heat equation

Let $u(x,y,t)$ be the temperature at the point (x,y).

A physical law says that u satisfies the heat equation

$$\boxed{\frac{\partial u}{\partial t} = \Delta u}$$

When the distribution of heat doesn't change anymore (steady-state), i.e., if $\frac{\partial u}{\partial t} = 0$, then we have the steady-state heat equation:

$$\Delta u = 0$$

(no heat source steady state)

So, for example, the solution

$$u(x,y) = \frac{1}{2}(x^2-y^2) + \frac{1}{2}$$

could be a function of temperature distribution on a unit disc.

Summary of this course

- Basic properties of analytic functions (Ch2-3)
  - *Cauchy-Riemann eq
  - examples: polynomials, e^z, $\sin z$, $\cos z$, functions defined by power series, $\log z$ (§8.2)

- *Line integrals (Ch4, Ch6, Ch8)
  - basic properties and M-L inequality: $\left|\int_C f(z)dz\right| \le \sup_{z\in C}|f(z)| \cdot \text{length}(C)$
  - closed curve thm & integral thm: domain is important!! (need simply connected)

- Applications of line integrals to analytic functions (Ch5-7)
  - Cauchy integral formula: $f(a) = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{z-a}dz$ (more generally, residue thm)
  - power series of analytic functions, uniqueness of analytic function
  - Liouville thm, MVT, Max-Modulus, open mapping thm, Morera thm, reflection principle, etc.
  - Schwarz' lemma —— applied to Riemann Mapping Thm (§14.2)

- Analytic functions with isolated singularities (§9.1 – §10.1)
  - classification of isolated singularities
  - Laurent expansion around an isolated singularity $\sum_{n=-\infty}^{\infty} a_n z^n$
  - residue: c_{-1} in Laurent expansion
  - *residue thm: $\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\cdot \text{Res}(f; z_k)$

    winding number

    isolated singularities inside the closed curve γ

- Applications of residue thm (§10.2 — Ch12)
  - argument principle, Rouché thm: count number of zeros
  - computation: integrals, sums (e.g. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ ~> zeta function)

- Conformal mapping (Ch13-14)
  - definitions and basic properties
  - Riemann Mapping Thm: any proper (i.e. $\neq \emptyset$, $\neq \mathbb{C}$) simply connected domains are conformally equivalent
  - examples of conformal maps: $az+b$, z^a, e^z, bilinear transformations $\frac{az+b}{cz+d}$, $ad-bc\neq 0$
  - construction of conformal maps

- Harmonic functions (Ch16)
  - relationship between harmonic functions and analytic functions
  - $\exists!$ problems of the eq. $\begin{cases} \Delta u = 0 \\ u|_{\partial U} = g \end{cases}$ ← given   cf. heat eq.

    $(U = D(0;1))$